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# Theory and Problems of Probability, Random Variables, and Random Processes

Hwei P. Hsu, Ph.D.

*Professor of Electrical Engineering*
*Fairleigh Dickinson University*

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**HWEI P. HSU** is Professor of Electrical Engineering at Fairleigh Dickinson University. He received his B.S. from National Taiwan University and M.S. and Ph.D. from Case Institute of Technology. He has published several books which include *Schaum's Outline of Analog and Digital Communications and Schaum's Outline of Signals and Systems.*

Schaum's Outline of Theory and Problems of

PROBABILITY, RANDOM VARIABLES, AND RANDOM PROCESSES

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# Preface

The purpose of this book is to provide an introduction to principles of probability, random variables, and random processes and their applications.

The book is designed for students in various disciplines of engineering, science, mathematics, and management. It may be used as a textbook and/or as a supplement to all current comparable texts. It should also be useful to those interested in the field for self-study. The book combines the advantages of both the textbook and the so-called review book. It provides the textual explanations of the textbook, and in the direct way characteristic of the review book, it gives hundreds of completely solved problems that use essential theory and techniques. Moreover, the solved problems are an integral part of the text. The background required to study the book is one year of calculus, elementary differential equations, matrix analysis, and some signal and system theory, including Fourier transforms.

I wish to thank Dr. Gordon Silverman for his invaluable suggestions and critical review of the manuscript. I also wish to express my appreciation to the editorial staff of the McGraw-Hill Schaum Series for their care, cooperation, and attention devoted to the preparation of the book. Finally, I thank my wife, Daisy, for her patience and encouragement.

HWEI P. HSU
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# Chapter 1

# Probability

## 1.1 INTRODUCTION

The study of probability stems from the analysis of certain games of chance, and it has found applications in most branches of science and engineering. In this chapter the basic concepts of probability theory are presented.

## 1.2 SAMPLE SPACE AND EVENTS

### A. Random Experiments:

In the study of probability, any process of observation is referred to as an *experiment*. The results of an observation are called the *outcomes* of the experiment. An experiment is called a *random experiment* if its outcome cannot be predicted. Typical examples of a random experiment are the roll of a die, the toss of a coin, drawing a card from a deck, or selecting a message signal for transmission from several messages.

### B. Sample Space:

The set of all possible outcomes of a random experiment is called the *sample space* (or *universal set*), and it is denoted by $S$. An element in $S$ is called a *sample point*. Each outcome of a random experiment corresponds to a sample point.

**EXAMPLE 1.1** Find the sample space for the experiment of tossing a coin (*a*) once and (*b*) twice.

(*a*) There are two possible outcomes, heads or tails. Thus

$$S = \{H, T\}$$

where $H$ and $T$ represent head and tail, respectively.

(*b*) There are four possible outcomes. They are pairs of heads and tails. Thus

$$S = \{HH, HT, TH, TT\}$$

**EXAMPLE 1.2** Find the sample space for the experiment of tossing a coin repeatedly and of counting the number of tosses required until the first head appears.

Clearly all possible outcomes for this experiment are the terms of the sequence 1, 2, 3, .... Thus

$$S = \{1, 2, 3, \ldots\}$$

Note that there are an infinite number of outcomes.

**EXAMPLE 1.3** Find the sample space for the experiment of measuring (in hours) the lifetime of a transistor.

Clearly all possible outcomes are all nonnegative real numbers. That is,

$$S = \{\tau: 0 \leq \tau \leq \infty\}$$

where $\tau$ represents the life of a transistor in hours.

Note that any particular experiment can often have many different sample spaces depending on the observation of interest (Probs. 1.1 and 1.2). A sample space $S$ is said to be *discrete* if it consists of a finite number of

sample points (as in Example 1.1) or countably infinite sample points (as in Example 1.2). A set is called *countable* if its elements can be placed in a one-to-one correspondence with the positive integers. A sample space $S$ is said to be *continuous* if the sample points constitute a continuum (as in Example 1.3).

### C. Events:

Since we have identified a sample space $S$ as the set of all possible outcomes of a random experiment, we will review some set notations in the following.

If $\zeta$ is an element of $S$ (or belongs to $S$), then we write

$$\zeta \in S$$

If $S$ is not an element of $S$ (or does not belong to $S$), then we write

$$\zeta \notin S$$

A set $A$ is called a *subset* of $B$, denoted by

$$A \subset B$$

if every element of $A$ is also an element of $B$. Any subset of the sample space $S$ is called an *event*. A sample point of $S$ is often referred to as an *elementary event*. Note that the sample space $S$ is the subset of itself, that is, $S \subset S$. Since $S$ is the set of all possible outcomes, it is often called the *certain event*.

**EXAMPLE 1.4** Consider the experiment of Example 1.2. Let $A$ be the event that the number of tosses required until the first head appears is even. Let $B$ be the event that the number of tosses required until the first head appears is odd. Let $C$ be the event that the number of tosses required until the first head appears is less than 5. Express events $A$, $B$, and $C$.

$$A = \{2, 4, 6, \ldots\}$$
$$B = \{1, 3, 5, \ldots\}$$
$$C = \{1, 2, 3, 4\}$$

## 1.3 ALGEBRA OF SETS

### A. Set Operations:

#### 1. *Equality:*

Two sets $A$ and $B$ are equal, denoted $A = B$, if and only if $A \subset B$ and $B \subset A$.

#### 2. *Complementation:*

Suppose $A \subset S$. The *complement* of set $A$, denoted $\bar{A}$, is the set containing all elements in $S$ but not in $A$.

$$\bar{A} = \{\zeta : \zeta \in S \text{ and } \zeta \notin A\}$$

#### 3. *Union:*

The *union* of sets $A$ and $B$, denoted $A \cup B$, is the set containing all elements in either $A$ or $B$ or both.

$$A \cup B = \{\zeta : \zeta \in A \text{ or } \zeta \in B\}$$

#### 4. *Intersection:*

The *intersection* of sets $A$ and $B$, denoted $A \cap B$, is the set containing all elements in both $A$ and $B$.

$$A \cap B = \{\zeta : \zeta \in A \text{ and } \zeta \in B\}$$

### 5. *Null Set:*

The set containing no element is called the *null set*, denoted $\varnothing$. Note that

$$\varnothing = \bar{S}$$

### 6. *Disjoint Sets:*

Two sets $A$ and $B$ are called *disjoint* or *mutually exclusive* if they contain no common element, that is, if $A \cap B = \varnothing$.

The definitions of the union and intersection of two sets can be extended to any finite number of sets as follows:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$
$$= \{\zeta: \zeta \in A_1 \text{ or } \zeta \in A_2 \text{ or } \cdots \zeta \in A_n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$
$$= \{\zeta: \zeta \in A_1 \text{ and } \zeta \in A_2 \text{ and } \cdots \zeta \in A_n\}$$

Note that these definitions can be extended to an infinite number of sets:

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$$
$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap A_3 \cap \cdots$$

In our definition of event, we state that every subset of $S$ is an event, including $S$ and the null set $\varnothing$. Then

$S$ = the certain event
$\varnothing$ = the impossible event

If $A$ and $B$ are events in $S$, then

$\bar{A}$ = the event that $A$ did not occur
$A \cup B$ = the event that either $A$ or $B$ or both occurred
$A \cap B$ = the event that both $A$ and $B$ occurred

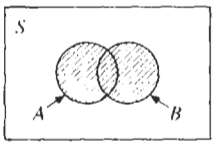
Similarly, if $A_1, A_2, \ldots, A_n$ are a sequence of events in $S$, then

$\bigcup_{i=1}^{n} A_i$ = the event that at least one of the $A_i$ occurred;

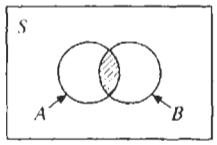
$\bigcap_{i=1}^{n} A_i$ = the event that all of the $A_i$ occurred.

## B. Venn Diagram:

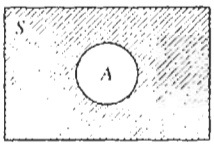
A graphical representation that is very useful for illustrating set operation is the Venn diagram. For instance, in the three Venn diagrams shown in Fig. 1-1, the shaded areas represent, respectively, the events $A \cup B$, $A \cap B$, and $\bar{A}$. The Venn diagram in Fig. 1-2 indicates that $B \subset A$ and the event $A \cap \bar{B}$ is shown as the shaded area.




(*a*) Shaded region: $A \cup B$

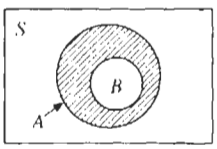



(*b*) Shaded region: $A \cap B$




(*c*) Shaded region: $\bar{A}$

**Fig. 1-1**




$B \subset A$

Shaded region: $A \cap \bar{B}$

**Fig. 1-2**

## C. Identities:

By the above set definitions or reference to Fig. 1-1, we obtain the following identities:

$$\bar{S} = \varnothing \tag{1.1}$$

$$\bar{\varnothing} = S \tag{1.2}$$

$$\bar{\bar{A}} = A \tag{1.3}$$

$$S \cup A = S \tag{1.4}$$

$$S \cap A = A \tag{1.5}$$

$$A \cup \bar{A} = S \tag{1.6}$$

$$A \cap \bar{A} = \varnothing \tag{1.7}$$

The union and intersection operations also satisfy the following laws:

***Commutative Laws:***

$$A \cup B = B \cup A \tag{1.8}$$

$$A \cap B = B \cap A \tag{1.9}$$

***Associative Laws:***

$$A \cup (B \cup C) = (A \cup B) \cup C \tag{1.10}$$

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{1.11}$$

***Distributive Laws:***

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{1.12}$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \tag{1.13}$$

***De Morgan's Laws:***

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \tag{1.14}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B} \tag{1.15}$$

These relations are verified by showing that any element that is contained in the set on the left side of the equality sign is also contained in the set on the right side, and vice versa. One way of showing this is by means of a Venn diagram (Prob. 1.13). The distributive laws can be extended as follows:

$$A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} (A \cap B_i) \tag{1.16}$$

$$A \cup \left(\bigcap_{i=1}^{n} B_i\right) = \bigcap_{i=1}^{n} (A \cup B_i) \tag{1.17}$$

Similarly, De Morgan's laws also can be extended as follows (Prob. 1.17):

$$\overline{\left(\bigcup_{i=1}^{n} A_i\right)} = \bigcap_{i=1}^{n} \bar{A}_i \tag{1.18}$$

$$\overline{\left(\bigcap_{i=1}^{n} A_i\right)} = \bigcup_{i=1}^{n} \bar{A}_i \tag{1.19}$$

## 1.4 THE NOTION AND AXIOMS OF PROBABILITY

An assignment of real numbers to the events defined in a sample space $S$ is known as the *probability measure*. Consider a random experiment with a sample space $S$, and let $A$ be a particular event defined in $S$.

### A. Relative Frequency Definition:

Suppose that the random experiment is repeated $n$ times. If event $A$ occurs $n(A)$ times, then the probability of event $A$, denoted $P(A)$, is defined as

$$P(A) = \lim_{n \to \infty} \frac{n(A)}{n} \tag{1.20}$$

where $n(A)/n$ is called the relative frequency of event $A$. Note that this limit may not exist, and in addition, there are many situations in which the concepts of repeatability may not be valid. It is clear that for any event $A$, the relative frequency of $A$ will have the following properties:

1. $0 \leq n(A)/n \leq 1$, where $n(A)/n = 0$ if $A$ occurs in none of the $n$ repeated trials and $n(A)/n = 1$ if $A$ occurs in all of the $n$ repeated trials.
2. If $A$ and $B$ are mutually exclusive events, then

$$n(A \cup B) = n(A) + n(B)$$

and

$$\frac{n(A \cup B)}{n} = \frac{n(A)}{n} + \frac{n(B)}{n}$$

**B. Axiomatic Definition:**

Let $S$ be a finite sample space and $A$ be an event in $S$. Then in the *axiomatic* definition, the probability $P(A)$ of the event $A$ is a real number assigned to $A$ which satisfies the following three *axioms*:

Axiom 1: $P(A) \geq 0$ (*1.21*)

Axiom 2: $P(S) = 1$ (*1.22*)

Axiom 3: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \varnothing$ (*1.23*)

If the sample space $S$ is not finite, then axiom 3 must be modified as follows:

Axiom 3′: If $A_1, A_2, \ldots$ is an infinite sequence of mutually exclusive events in $S$ ($A_i \cap A_j = \varnothing$ for $i \neq j$), then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \tag{1.24}$$

These axioms satisfy our intuitive notion of probability measure obtained from the notion of relative frequency.

**C. Elementary Properties of Probability:**

By using the above axioms, the following useful properties of probability can be obtained:

1. $P(\bar{A}) = 1 - P(A)$ (*1.25*)
2. $P(\varnothing) = 0$ (*1.26*)
3. $P(A) \leq P(B)$ if $A \subset B$ (*1.27*)
4. $P(A) \leq 1$ (*1.28*)
5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (*1.29*)
6. If $A_1, A_2, \ldots, A_n$ are $n$ arbitrary events in $S$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) - \cdots (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n) \tag{1.30}$$

where the sum of the second term is over all distinct pairs of events, that of the third term is over all distinct triples of events, and so forth.

7. If $A_1, A_2, \ldots, A_n$ is a finite sequence of mutually exclusive events in $S$ ($A_i \cap A_j = \varnothing$ for $i \neq j$), then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \tag{1.31}$$

and a similar equality holds for any subcollection of the events.

Note that property 4 can be easily derived from axiom 2 and property 3. Since $A \subset S$, we have

$$P(A) \leq P(S) = 1$$

Thus, combining with axiom 1, we obtain

$$0 \le P(A) \le 1 \tag{1.32}$$

Property 5 implies that

$$P(A \cup B) \le P(A) + P(B) \tag{1.33}$$

since $P(A \cap B) \ge 0$ by axiom 1.

## 1.5 EQUALLY LIKELY EVENTS

### A. Finite Sample Space:

Consider a finite sample space $S$ with $n$ finite elements

$$S = \{\zeta_1, \zeta_2, \ldots, \zeta_n\}$$

where $\zeta_i$'s are elementary events. Let $P(\zeta_i) = p_i$. Then

1. $0 \le p_i \le 1 \qquad i = 1, 2, \ldots, n$ (1.34)

2. $\sum_{i=1}^{n} p_i = p_1 + p_2 + \cdots + p_n = 1$ (1.35)

3. If $A = \bigcup_{i \in I} \zeta_i$, where $I$ is a collection of subscripts, then

$$P(A) = \sum_{\zeta_i \in A} P(\zeta_i) = \sum_{i \in I} p_i \tag{1.36}$$

### B. Equally Likely Events:

When all elementary events $\zeta_i$ $(i = 1, 2, \ldots, n)$ are equally likely, that is,

$$p_1 = p_2 = \cdots = p_n$$

then from Eq. (1.35), we have

$$p_i = \frac{1}{n} \qquad i = 1, 2, \ldots, n \tag{1.37}$$

and

$$P(A) = \frac{n(A)}{n} \tag{1.38}$$

where $n(A)$ is the number of outcomes belonging to event $A$ and $n$ is the number of sample points in $S$.

## 1.6 CONDITIONAL PROBABILITY

### A. Definition:

The *conditional probability* of an event $A$ given event $B$, denoted by $P(A \mid B)$, is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0 \tag{1.39}$$

where $P(A \cap B)$ is the joint probability of $A$ and $B$. Similarly,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} \qquad P(A) > 0 \tag{1.40}$$

is the conditional probability of an event $B$ given event $A$. From Eqs. (*1.39*) and (*1.40*), we have

$$P(A \cap B) = P(A \mid B)P(B) = P(B \mid A)P(A) \tag{1.41}$$

Equation (*1.41*) is often quite useful in computing the joint probability of events.

**B. Bayes' Rule:**

From Eq. (*1.41*) we can obtain the following *Bayes' rule*:

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \tag{1.42}$$

## 1.7 TOTAL PROBABILITY

The events $A_1, A_2, \ldots, A_n$ are called *mutually exclusive and exhaustive* if

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n = S \qquad \text{and} \qquad A_i \cap A_j = \varnothing \qquad i \neq j \tag{1.43}$$

Let $B$ be any event in $S$. Then

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i) \tag{1.44}$$

which is known as the *total probability* of event $B$ (Prob. 1.47). Let $A = A_i$ in Eq. (*1.42*); then, using Eq. (*1.44*), we obtain

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{\sum_{i=1}^{n} P(B \mid A_i)P(A_i)} \tag{1.45}$$

Note that the terms on the right-hand side are all conditioned on events $A_i$, while the term on the left is conditioned on $B$. Equation (*1.45*) is sometimes referred to as *Bayes' theorem.*

## 1.8 INDEPENDENT EVENTS

Two events $A$ and $B$ are said to be (*statistically*) *independent* if and only if

$$P(A \cap B) = P(A)P(B) \tag{1.46}$$

It follows immediately that if $A$ and $B$ are independent, then by Eqs. (*1.39*) and (*1.40*),

$$P(A \mid B) = P(A) \qquad \text{and} \qquad P(B \mid A) = P(B) \tag{1.47}$$

If two events $A$ and $B$ are independent, then it can be shown that $A$ and $\bar{B}$ are also independent; that is (Prob. 1.53),

$$P(A \cap \bar{B}) = P(A)P(\bar{B}) \tag{1.48}$$

Then

$$P(A \mid \bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = P(A) \tag{1.49}$$

Thus, if $A$ is independent of $B$, then the probability of $A$'s occurrence is unchanged by information as to whether or not $B$ has occurred. Three events $A$, $B$, $C$ are said to be independent if and only if

$$\begin{aligned} P(A \cap B \cap C) &= P(A)P(B)P(C) \\ P(A \cap B) &= P(A)P(B) \\ P(A \cap C) &= P(A)P(C) \\ P(B \cap C) &= P(B)P(C) \end{aligned} \tag{1.50}$$

We may also extend the definition of independence to more than three events. The events $A_1, A_2, \ldots, A_n$ are independent if and only if for every subset $\{A_{i_1}, A_{i_2}, \ldots, A_{i_k}\}$ $(2 \leq k \leq n)$ of these events,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}) \tag{1.51}$$

Finally, we define an infinite set of events to be independent if and only if every finite subset of these events is independent.

To distinguish between the mutual exclusiveness (or disjointness) and independence of a collection of events we summarize as follows:

1. If $\{A_i, i = 1, 2, \ldots, n\}$ is a sequence of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \tag{1.52}$$

2. If $\{A_i, i = 1, 2, \ldots, n\}$ is a sequence of independent events, then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i) \tag{1.53}$$

and a similar equality holds for any subcollection of the events.

## Solved Problems

### SAMPLE SPACE AND EVENTS

**1.1.** Consider a random experiment of tossing a coin three times.

(*a*) Find the sample space $S_1$ if we wish to observe the exact sequences of heads and tails obtained.

(*b*) Find the sample space $S_2$ if we wish to observe the number of heads in the three tosses.

(*a*) The sampling space $S_1$ is given by

$$S_1 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

where, for example, $HTH$ indicates a head on the first and third throws and a tail on the second throw. There are eight sample points in $S_1$.

(*b*) The sampling space $S_2$ is given by

$$S_2 = \{0, 1, 2, 3\}$$

where, for example, the outcome 2 indicates that two heads were obtained in the three tosses. The sample space $S_2$ contains four sample points.

**1.2.** Consider an experiment of drawing two cards at random from a bag containing four cards marked with the integers 1 through 4.

(*a*) Find the sample space $S_1$ of the experiment if the first card is replaced before the second is drawn.

(*b*) Find the sample space $S_2$ of the experiment if the first card is not replaced.

(*a*) The sample space $S_1$ contains 16 ordered pairs $(i, j)$, $1 \leq i \leq 4$, $1 \leq j \leq 4$, where the first number indicates the first number drawn. Thus,

$$S_1 = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{Bmatrix}$$

(*b*) The sample space $S_2$ contains 12 ordered pairs $(i, j)$, $i \neq j$, $1 \leq i \leq 4$, $1 \leq j \leq 4$, where the first number indicates the first number drawn. Thus,

$$S_2 = \begin{Bmatrix} (1,2) & (1,3) & (1,4) \\ (2,1) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,4) \\ (4,1) & (4,2) & (4,3) \end{Bmatrix}$$

**1.3.** An experiment consists of rolling a die until a 6 is obtained.

(*a*) Find the sample space $S_1$ if we are interested in all possibilities.

(*b*) Find the sample space $S_2$ if we are interested in the number of throws needed to get a 6.

(*a*) The sample space $S_1$ would be

$$S_1 = \{6,\ 16,\ 26,\ 36,\ 46,\ 56,\ 116,\ 126,\ 136,\ 146,\ 156, \ldots\}$$

where the first line indicates that a 6 is obtained in one throw, the second line indicates that a 6 is obtained in two throws, and so forth.

(*b*) In this case, the sample space $S_2$ is

$$S_2 = \{i: i \geq 1\} = \{1, 2, 3, \ldots\}$$

where $i$ is an integer representing the number of throws needed to get a 6.

**1.4.** Find the sample space for the experiment consisting of measurement of the voltage output $v$ from a transducer, the maximum and minimum of which are $+5$ and $-5$ volts, respectively.

A suitable sample space for this experiment would be

$$S = \{v: -5 \leq v \leq 5\}$$

**1.5.** An experiment consists of tossing two dice.

(*a*) Find the sample space $S$.

(*b*) Find the event $A$ that the sum of the dots on the dice equals 7.

(*c*) Find the event $B$ that the sum of the dots on the dice is greater than 10.

(*d*) Find the event $C$ that the sum of the dots on the dice is greater than 12.

(*a*) For this experiment, the sample space $S$ consists of 36 points (Fig. 1-3):

$$S = \{(i, j): i, j = 1, 2, 3, 4, 5, 6\}$$

where $i$ represents the number of dots appearing on one die and $j$ represents the number of dots appearing on the other die.

(*b*) The event $A$ consists of 6 points (see Fig. 1-3):

$$A = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$$

(*c*) The event $B$ consists of 3 points (see Fig. 1-3):

$$B = \{(5, 6), (6, 5), (6, 6)\}$$

(*d*) The event $C$ is an impossible event, that is, $C = \varnothing$.

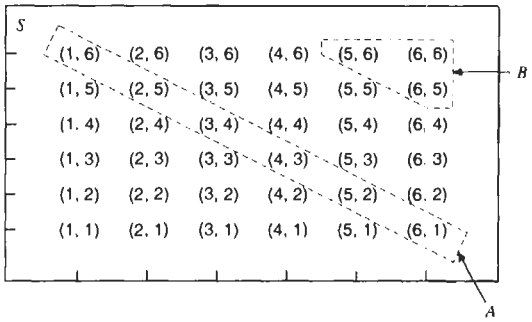



**Fig. 1-3**

**1.6.** An automobile dealer offers vehicles with the following options:

(*a*) With or without automatic transmission

(*b*) With or without air-conditioning

(*c*) With one of two choices of a stereo system

(*d*) With one of three exterior colors

If the sample space consists of the set of all possible vehicle types, what is the number of outcomes in the sample space?

The tree diagram for the different types of vehicles is shown in Fig. 1-4. From Fig. 1-4 we see that the number of sample points in $S$ is $2 \times 2 \times 2 \times 3 = 24$.

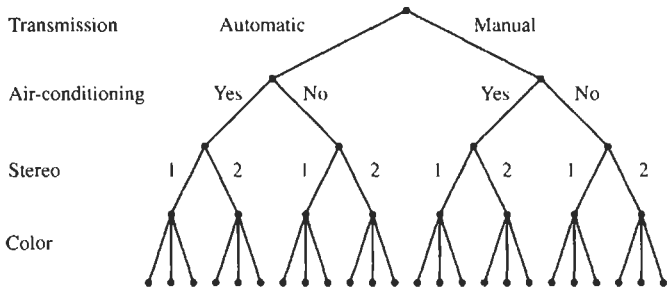



**Fig. 1-4**

**1.7.** State every possible event in the sample space $S = \{a, b, c, d\}$.

There are $2^4 = 16$ possible events in $S$. They are $\varnothing$; $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$; $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$; $\{a, b, c\}$, $(a, b, d)$, $(a, c, d\}$, $\{b, c, d\}$; $S = \{a, b, c, d\}$.

**1.8.** How many events are there in a sample space $S$ with $n$ elementary events?

Let $S = \{s_1, s_2, \ldots, s_n\}$. Let $\Omega$ be the family of all subsets of $S$. ($\Omega$ is sometimes referred to as the *power set* of $S$.) Let $S_i$ be the set consisting of two statements, that is,

$$S_i = \{\text{Yes, the } s_i \text{ is in; No, the } s_i \text{ is not in}\}$$

Then $\Omega$ can be represented as the Cartesian product

$$\begin{aligned}\Omega &= S_1 \times S_2 \times \cdots \times S_n \\ &= \{(s_1, s_2, \ldots, s_n) : s_i \in S_i \text{ for } i = 1, 2, \ldots, n\}\end{aligned}$$

Since each subset of $S$ can be uniquely characterized by an element in the above Cartesian product, we obtain the number of elements in $\Omega$ by

$$n(\Omega) = n(S_1)n(S_2)\cdots n(S_n) = 2^n$$

where $n(S_i)$ = number of elements in $S_i = 2$.

An alternative way of finding $n(\Omega)$ is by the following summation:

$$n(\Omega) = \sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!}$$

The proof that the last sum is equal to $2^n$ is not easy.

## ALGEBRA OF SETS

**1.9.** Consider the experiment of Example 1.2. We define the events

$$A = \{k: k \text{ is odd}\}$$
$$B = \{k: 4 \le k \le 7\}$$
$$C = \{k: 1 \le k \le 10\}$$

where $k$ is the number of tosses required until the first $H$ (head) appears. Determine the events $\bar{A}$, $\bar{B}$, $\bar{C}$, $A \cup B$, $B \cup C$, $A \cap B$, $A \cap C$, $B \cap C$, and $\bar{A} \cap B$.

$$\bar{A} = \{k: k \text{ is even}\} = \{2, 4, 6, \ldots\}$$
$$\bar{B} = \{k: k = 1, 2, 3 \text{ or } k \ge 8\}$$
$$\bar{C} = \{k: k \ge 11\}$$
$$A \cup B = \{k: k \text{ is odd or } k = 4, 6\}$$
$$B \cup C = C$$
$$A \cap B = \{5, 7\}$$
$$A \cap C = \{1, 3, 5, 7, 9\}$$
$$B \cap C = B$$
$$\bar{A} \cap B = \{4, 6\}$$

**1.10.** The sample space of an experiment is the real line expressed as

$$S = \{v: -\infty < v < \infty\}$$

(*a*) Consider the events

$$A_1 = \{v: 0 \le v < \tfrac{1}{2}\}$$
$$A_2 = \{v: \tfrac{1}{2} \le v < \tfrac{3}{4}\}$$
$$A_i = \left\{v: 1 - \frac{1}{2^{i-1}} \le v < 1 - \frac{1}{2^i}\right\}$$

Determine the events

$$\bigcup_{i=1}^{\infty} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} A_i$$

(*b*) Consider the events

$$B_1 = \{v: v \le \tfrac{1}{2}\}$$
$$B_2 = \{v: v \le \tfrac{1}{4}\}$$
$$\vdots$$
$$B_i = \left\{v: v \le \frac{1}{2^i}\right\}$$

Determine the events

$$\bigcup_{i=1}^{\infty} B_i \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} B_i$$

(*a*) It is clear that

$$\bigcup_{i=1}^{\infty} A_i = \{v: 0 \le v < 1\}$$

Noting that the $A_i$'s are mutually exclusive, we have

$$\bigcap_{i=1}^{\infty} A_i = \varnothing$$

(*b*) Noting that $B_1 \supset B_2 \supset \cdots \supset B_i \supset \cdots$, we have

$$\bigcup_{i=1}^{\infty} B_i = B_1 = \{v: v \le \tfrac{1}{2}\} \qquad \text{and} \qquad \bigcap_{i=1}^{\infty} B_i = \{v: v \le 0\}$$

**1.11.** Consider the switching networks shown in Fig. 1-5. Let $A_1$, $A_2$, and $A_3$ denote the events that the switches $s_1$, $s_2$, and $s_3$ are closed, respectively. Let $A_{ab}$ denote the event that there is a closed path between terminals $a$ and $b$. Express $A_{ab}$ in terms of $A_1$, $A_2$, and $A_3$ for each of the networks shown.

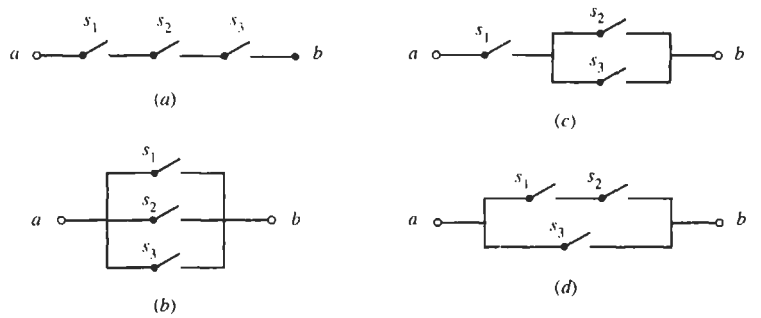



**Fig. 1-5**

(*a*) From Fig. 1-5(*a*), we see that there is a closed path between $a$ and $b$ only if all switches $s_1$, $s_2$, and $s_3$ are closed. Thus,

$$A_{ab} = A_1 \cap A_2 \cap A_3$$

(*b*) From Fig. 1-5(*b*), we see that there is a closed path between $a$ and $b$ if at least one switch is closed. Thus,

$$A_{ab} = A_1 \cup A_2 \cup A_3$$

(*c*) From Fig. 1-5(*c*), we see that there is a closed path between $a$ and $b$ if $s_1$ and either $s_2$ or $s_3$ are closed. Thus,

$$A_{ab} = A_1 \cap (A_2 \cup A_3)$$

Using the distributive law (1.12), we have

$$A_{ab} = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

which indicates that there is a closed path between $a$ and $b$ if $s_1$ and $s_2$ or $s_1$ and $s_3$ are closed.

(*d*) From Fig. 1-5(*d*), we see that there is a closed path between $a$ and $b$ if either $s_1$ and $s_2$ are closed or $s_3$ is closed. Thus

$$A_{ab} = (A_1 \cap A_2) \cup A_3$$

**1.12.** Verify the distributive law (*1.12*).

Let $s \in [A \cap (B \cup C)]$. Then $s \in A$ and $s \in (B \cup C)$. This means either that $s \in A$ and $s \in B$ or that $s \in A$ and $s \in C$; that is, $s \in (A \cap B)$ or $s \in (A \cap C)$. Therefore,

$$A \cap (B \cup C) \subset [(A \cap B) \cup (A \cap C)]$$

Next, let $s \in [(A \cap B) \cup (A \cap C)]$. Then $s \in A$ and $s \in B$ or $s \in A$ and $s \in C$. Thus $s \in A$ and ($s \in B$ or $s \in C$). Thus,

$$[(A \cap B) \cup (A \cap C)] \subset A \cap (B \cup C)$$

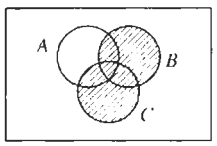
Thus, by the definition of equality, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

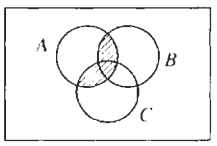
**1.13.** Using a Venn diagram, repeat Prob. 1.12.

Figure 1-6 shows the sequence of relevant Venn diagrams. Comparing Fig. 1-6(*b*) and 1-6(*e*), we conclude that

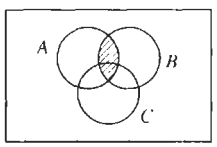
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



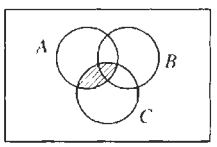

(*a*) Shaded region: $B \cup C$



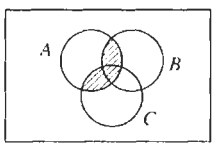

(*b*) Shaded region: $A \cap (B \cup C)$




(*c*) Shaded region: $A \cap B$




(*d*) Shaded region: $A \cap C$




(*e*) Shaded region: $(A \cap B) \cup (A \cap C)$

**Fig. 1-6**

**1.14.** Let $A$ and $B$ be arbitrary events. Show that $A \subset B$ if and only if $A \cap B = A$.

"If" part: We show that if $A \cap B = A$, then $A \subset B$. Let $s \in A$. Then $s \in (A \cap B)$, since $A = A \cap B$. Then by the definition of intersection, $s \in B$. Therefore, $A \subset B$.

"Only if" part: We show that if $A \subset B$, then $A \cap B = A$. Note that from the definition of the intersection, $(A \cap B) \subset A$. Suppose $s \in A$. If $A \subset B$, then $s \in B$. So $s \in A$ and $s \in B$; that is, $s \in (A \cap B)$. Therefore, it follows that $A \subset (A \cap B)$. Hence, $A = A \cap B$. This completes the proof.

**1.15.** Let $A$ be an arbitrary event in $S$ and let $\varnothing$ be the null event. Show that

(*a*) $A \cup \varnothing = A$ (*1.54*)

(*b*) $A \cap \varnothing = \varnothing$ (*1.55*)

(*a*) $A \cup \varnothing = \{s: s \in A \text{ or } s \in \varnothing\}$
But, by definition, there are no $s \in \varnothing$. Thus,

$$A \cup \varnothing = \{s: s \in A\} = A$$

(*b*) $A \cap \varnothing = \{s: s \in A \text{ and } s \in \varnothing\}$
But, since there are no $s \in \varnothing$, there cannot be an $s$ such that $s \in A$ and $s \in \varnothing$. Thus,

$$A \cap \varnothing = \varnothing$$

Note that Eq. (*1.55*) shows that $\varnothing$ is mutually exclusive with every other event and including with itself.

**1.16.** Show that the null (or empty) set $\varnothing$ is a subset of every set $A$.

From the definition of intersection, it follows that

$$(A \cap B) \subset A \quad \text{and} \quad (A \cap B) \subset B \tag{1.56}$$

for any pair of events, whether they are mutually exclusive or not. If $A$ and $B$ are mutually exclusive events, that is, $A \cap B = \varnothing$, then by Eq. (*1.56*) we obtain

$$\varnothing \subset A \quad \text{and} \quad \varnothing \subset B \tag{1.57}$$

Therefore, for any event $A$,

$$\varnothing \subset A \tag{1.58}$$

that is, $\varnothing$ is a subset of every set $A$.

**1.17.** Verify Eqs. (*1.18*) and (*1.19*).

(*a*) Suppose first that $s \in \overline{\left(\bigcup_{i=1}^{n} A_i\right)}$; then $s \notin \left(\bigcup_{i=1}^{n} A_i\right)$.

That is, if $s$ is not contained in any of the events $A_i$, $i = 1, 2, \ldots, n$, then $s$ is contained in $\bar{A}_i$ for all $i = 1, 2, \ldots, n$. Thus

$$s \in \bigcap_{i=1}^{n} \bar{A}_i$$

Next, we assume that

$$s \in \bigcap_{i=1}^{n} \bar{A}_i$$

Then $s$ is contained in $\bar{A}_i$ for all $i = 1, 2, \ldots, n$, which means that $s$ is not contained in $A_i$ for any $i = 1, 2, \ldots, n$, implying that

$$s \notin \bigcup_{i=1}^{n} A_i$$

Thus,

$$s \in \overline{\left(\bigcup_{i=1}^{n} A_i\right)}$$

This proves Eq. (*1.18*).

(*b*) Using Eqs. (*1.18*) and (*1.3*), we have

$$\overline{\left(\bigcup_{i=1}^{n} \bar{A}_i\right)} = \bigcap_{i=1}^{n} A_i$$

Taking complements of both sides of the above yields

$$\bigcup_{i=1}^{n} \bar{A}_i = \overline{\left(\bigcap_{i=1}^{n} A_i\right)}$$

which is Eq. (*1.19*).

## THE NOTION AND AXIOMS OF PROBABILITY

**1.18.** Using the axioms of probability, prove Eq. (*1.25*).

We have

$$S = A \cup \bar{A} \qquad \text{and} \qquad A \cap \bar{A} = \varnothing$$

Thus, by axioms 2 and 3, it follows that

$$P(S) = 1 = P(A) + P(\bar{A})$$

from which we obtain

$$P(\bar{A}) = 1 - P(A)$$

**1.19.** Verify Eq. (*1.26*).

From Eq. (*1.25*), we have

$$P(A) = 1 - P(\bar{A})$$

Let $A = \varnothing$. Then, by Eq. (*1.2*), $\bar{A} = \varnothing = S$, and by axiom 2 we obtain

$$P(\varnothing) = 1 - P(S) = 1 - 1 = 0$$

**1.20.** Verify Eq. (*1.27*).

Let $A \subset B$. Then from the Venn diagram shown in Fig. 1-7, we see that

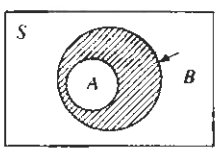
$$B = A \cup (\bar{A} \cap B) \qquad \text{and} \qquad A \cap (\bar{A} \cap B) = \varnothing \tag{1.59}$$

Hence, from axiom 3,

$$P(B) = P(A) + P(\bar{A} \cap B)$$

However, by axiom 1, $P(\bar{A} \cap B) \geq 0$. Thus, we conclude that

$$P(A) \leq P(B) \qquad \text{if } A \subset B$$




Shaded region: $\bar{A} \cap B$

**Fig. 1-7**

**1.21.** Verify Eq. (*1.29*).

From the Venn diagram of Fig. 1-8, each of the sets $A \cup B$ and $B$ can be represented, respectively, as a union of mutually exclusive sets as follows:

$$A \cup B = A \cup (\bar{A} \cap B) \qquad \text{and} \qquad B = (A \cap B) \cup (\bar{A} \cap B)$$

Thus, by axiom 3,

$$P(A \cup B) = P(A) + P(\bar{A} \cap B) \tag{1.60}$$

and

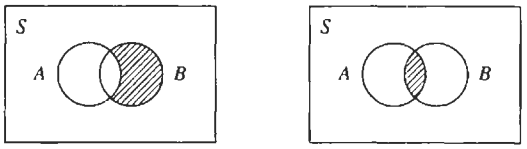
$$P(B) = P(A \cap B) + P(\bar{A} \cap B) \tag{1.61}$$

From Eq. (*1.61*), we have

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) \tag{1.62}$$

Substituting Eq. (*1.62*) into Eq. (*1.60*), we obtain

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$




Shaded region: $\bar{A} \cap B$ Shaded region: $A \cap B$

**Fig. 1-8**

**1.22.** Let $P(A) = 0.9$ and $P(B) = 0.8$. Show that $P(A \cap B) \geq 0.7$.

From Eq. (*1.29*), we have

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

By Eq. (*1.32*), $0 \leq P(A \cup B) \leq 1$. Hence

$$P(A \cap B) \geq P(A) + P(B) - 1 \tag{1.63}$$

Substituting the given values of $P(A)$ and $P(B)$ in Eq. (*1.63*), we get

$$P(A \cap B) \geq 0.9 + 0.8 - 1 = 0.7$$

Equation (1.63) is known as *Bonferroni's inequality*.

**1.23.** Show that

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) \tag{1.64}$$

From the Venn diagram of Fig. 1-9, we see that

$$A = (A \cap B) \cup (A \cap \bar{B}) \quad \text{and} \quad (A \cap B) \cap (A \cap \bar{B}) = \varnothing \tag{1.65}$$

Thus, by axiom 3, we have

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

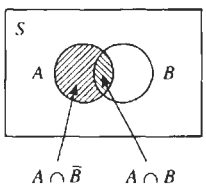



**Fig. 1-9**

**1.24.** Given that $P(A) = 0.9$, $P(B) = 0.8$, and $P(A \cap B) = 0.75$, find (*a*) $P(A \cup B)$; (*b*) $P(A \cap \bar{B})$; and (*c*) $P(\bar{A} \cap \bar{B})$.

(*a*) By Eq. (*1.29*), we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.9 + 0.8 - 0.75 = 0.95$$

(*b*) By Eq. (*1.64*) (Prob. 1.23), we have

$$P(A \cap \bar{B}) = P(A) - P(A \cap B) = 0.9 - 0.75 = 0.15$$

(*c*) By De Morgan's law, Eq. (*1.14*), and Eq. (*1.25*) and using the result from part (*a*), we get

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - 0.95 = 0.05$$

**1.25.** For any three events $A_1$, $A_2$, and $A_3$, show that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \tag{1.66}$$

Let $B = A_2 \cup A_3$. By Eq. (*1.29*), we have

$$P(A_1 \cup B) = P(A_1) + P(B) - P(A_1 \cap B) \tag{1.67}$$

Using distributive law (*1.12*), we have

$$A_1 \cap B = A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

Applying Eq. (*1.29*) to the above event, we obtain

$$\begin{aligned} P(A_1 \cap B) &= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P[(A_1 \cap A_2) \cap (A_1 \cap A_3)] \\ &= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3) \end{aligned} \tag{1.68}$$

Applying Eq. (*1.29*) to the set $B = A_2 \cup A_3$, we have

$$P(B) = P(A_2 \cup A_3) = P(A_2) + P(A_3) - P(A_2 \cap A_3) \tag{1.69}$$

Substituting Eqs. (*1.69*) and (*1.68*) into Eq. (*1.67*), we get

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

**1.26.** Prove that

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i) \tag{1.70}$$

which is known as *Boole's inequality*.

We will prove Eq. (*1.70*) by induction. Suppose Eq. (*1.70*) is true for $n = k$.

$$P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$$

Then

$$\begin{aligned} P\left(\bigcup_{i=1}^{k+1} A_i\right) &= P\left[\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right] \\ &\le P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) \qquad \text{[by Eq. (1.33)]} \\ &\le \sum_{i=1}^{k} P(A_i) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i) \end{aligned}$$

Thus Eq. (*1.70*) is also true for $n = k + 1$. By Eq. (*1.33*), Eq. (*1.70*) is true for $n = 2$. Thus, Eq. (*1.70*) is true for $n \ge 2$.

**1.27.** Verify Eq. (*1.31*).

Again we prove it by induction. Suppose Eq. (*1.31*) is true for $n = k$.

$$P\left(\bigcup_{i=1}^{k} A_i\right) = \sum_{i=1}^{k} P(A_i)$$

Then

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left[\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right]$$

Using the distributive law (*1.16*), we have

$$\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1} = \bigcup_{i=1}^{k} (A_i \cap A_{k+1}) = \bigcup_{i=1}^{k} \varnothing = \varnothing$$

since $A_i \cap A_j = \varnothing$ for $i \neq j$. Thus, by axiom 3, we have

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i)$$

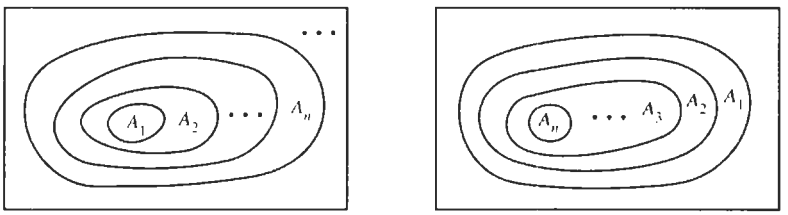
which indicates that Eq. (*1.31*) is also true for $n = k + 1$. By axiom 3, Eq. (*1.31*) is true for $n = 2$. Thus, it is true for $n \geq 2$.

**1.28.** A sequence of events $\{A_n, n \geq 1\}$ is said to be an *increasing sequence* if [Fig. 1-10(*a*)]

$$A_1 \subset A_2 \subset \cdots \subset A_k \subset A_{k+1} \subset \cdots \tag{1.71a}$$

whereas it is said to be a *decreasing sequence* if [Fig. 1-10(*b*)]

$$A_1 \supset A_2 \supset \cdots \supset A_k \supset A_{k+1} \supset \cdots \tag{1.71b}$$




(*a*) (*b*)

**Fig. 1-10**

If $\{A_n, n \geq 1\}$ is an increasing sequence of events, we define a new event $A_\infty$ by

$$A_\infty = \lim_{n \to \infty} A_n = \bigcup_{i=1}^{\infty} A_i \tag{1.72}$$

Similarly, if $\{A_n, n \geq 1\}$ is a decreasing sequence of events, we define a new event $A_\infty$ by

$$A_\infty = \lim_{n \to \infty} A_n = \bigcap_{i=1}^{\infty} A_i \tag{1.73}$$

Show that if $\{A_n, n \geq 1\}$ is either an increasing or a decreasing sequence of events, then

$$\lim_{n \to \infty} P(A_n) = P(A_\infty) \tag{1.74}$$

which is known as the *continuity theorem of probability.*

If $\{A_n, n \geq 1\}$ is an increasing sequence of events, then by definition

$$\bigcup_{i=1}^{n-1} A_i = A_{n-1}$$

Now, we define the events $B_n$, $n \geq 1$, by

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \cap \bar{A}_1 \\ &\vdots \\ B_n &= A_n \cap \bar{A}_{n-1} \end{aligned}$$

Thus, $B_n$ consists of those elements in $A_n$ that are not in any of the earlier $A_k$, $k < n$. From the Venn diagram shown in Fig. 1-11, it is seen that $B_n$ are mutually exclusive events such that

$$\bigcup_{i=1}^{n} B_i = \bigcup_{i=1}^{n} A_i \text{ for all } n \geq 1, \text{ and } \bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i = A_\infty$$

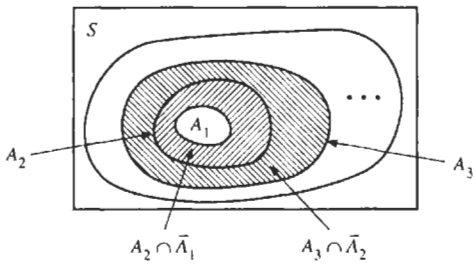



**Fig. 1-11**

Thus, using axiom 3′, we have

$$P(A_\infty) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} P(B_i)$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} P(B_i) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right)$$
$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) = \lim_{n\to\infty} P(A_n) \tag{1.75}$$

Next, if $\{A_n,\ n \geq 1\}$ is a decreasing sequence, then $\{\bar{A}_n,\ n \geq 1\}$ is an increasing sequence. Hence, by Eq. (*1.75*), we have

$$P\left(\bigcup_{i=1}^{\infty} \bar{A}_i\right) = \lim_{n\to\infty} P(\bar{A}_n)$$

From Eq. (*1.19*),

$$\bigcup_{i=1}^{\infty} \bar{A}_i = \overline{\left(\bigcap_{i=1}^{\infty} A_i\right)}$$

Thus,
$$P\left[\overline{\left(\bigcap_{i=1}^{\infty} A_i\right)}\right] = \lim_{n\to\infty} P(\bar{A}_n) \tag{1.76}$$

Using Eq. (*1.25*), Eq. (*1.76*) reduces to

$$1 - P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} [1 - P(A_n)] = 1 - \lim_{n\to\infty} P(A_n)$$

Thus,
$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = P(A_\infty) = \lim_{n\to\infty} P(A_n) \tag{1.77}$$

Combining Eqs. (*1.75*) and (*1.77*), we obtain Eq. (*1.74*).

## EQUALLY LIKELY EVENTS

**1.29.** Consider a telegraph source generating two symbols, dots and dashes. We observed that the dots were twice as likely to occur as the dashes. Find the probabilities of the dot's occurring and the dash's occurring.

From the observation, we have

$$P(\text{dot}) = 2P(\text{dash})$$

Then, by Eq. (*1.35*),

$$P(\text{dot}) + P(\text{dash}) = 3P(\text{dash}) = 1$$

Thus, $P(\text{dash}) = \frac{1}{3}$ and $P(\text{dot}) = \frac{2}{3}$

**1.30.** The sample space $S$ of a random experiment is given by

$$S = \{a, b, c, d\}$$

with probabilities $P(a) = 0.2$, $P(b) = 0.3$, $P(c) = 0.4$, and $P(d) = 0.1$. Let $A$ denote the event $\{a, b\}$, and $B$ the event $\{b, c, d\}$. Determine the following probabilities: (*a*) $P(A)$; (*b*) $P(B)$; (*c*) $P(\bar{A})$; (*d*) $P(A \cup B)$; and (*e*) $P(A \cap B)$.

Using Eq. (*1.36*), we obtain

(*a*) $P(A) = P(a) + P(b) = 0.2 + 0.3 = 0.5$

(*b*) $P(B) = P(b) + P(c) + P(d) = 0.3 + 0.4 + 0.1 = 0.8$

(*c*) $\bar{A} = \{c, d\}$; $P(\bar{A}) = P(c) + P(d) = 0.4 + 0.1 = 0.5$

(*d*) $A \cup B = \{a, b, c, d\} = S$; $P(A \cup B) = P(S) = 1$

(*e*) $A \cap B = \{b\}$; $P(A \cap B) = P(b) = 0.3$

**1.31.** An experiment consists of observing the sum of the dice when two fair dice are thrown (Prob. 1.5). Find (*a*) the probability that the sum is 7 and (*b*) the probability that the sum is greater than 10.

(*a*) Let $\zeta_{ij}$ denote the elementary event (sampling point) consisting of the following outcome: $\zeta_{ij} = (i, j)$, where $i$ represents the number appearing on one die and $j$ represents the number appearing on the other die. Since the dice are fair, all the outcomes are equally likely. So $P(\zeta_{ij}) = \frac{1}{36}$. Let $A$ denote the event that the sum is 7. Since the events $\zeta_{ij}$ are mutually exclusive and from Fig. 1-3 (Prob. 1.5), we have

$$\begin{aligned} P(A) &= P(\zeta_{16} \cup \zeta_{25} \cup \zeta_{34} \cup \zeta_{43} \cup \zeta_{52} \cup \zeta_{61}) \\ &= P(\zeta_{16}) + P(\zeta_{25}) + P(\zeta_{34}) + P(\zeta_{43}) + P(\zeta_{52}) + P(\zeta_{61}) \\ &= 6(\tfrac{1}{36}) = \tfrac{1}{6} \end{aligned}$$

(*b*) Let $B$ denote the event that the sum is greater than 10. Then from Fig. 1-3, we obtain

$$\begin{aligned} P(B) &= P(\zeta_{56} \cup \zeta_{65} \cup \zeta_{66}) = P(\zeta_{56}) + P(\zeta_{65}) + P(\zeta_{66}) \\ &= 3(\tfrac{1}{36}) = \tfrac{1}{12} \end{aligned}$$

**1.32.** There are $n$ persons in a room.

(*a*) What is the probability that at least two persons have the same birthday?

(*b*) Calculate this probability for $n = 50$.

(*c*) How large need $n$ be for this probability to be greater than 0.5?

(*a*) As each person can have his or her birthday on any one of 365 days (ignoring the possibility of February 29), there are a total of $(365)^n$ possible outcomes. Let $A$ be the event that no two persons have the same birthday. Then the number of outcomes belonging to $A$ is

$$n(A) = (365)(364) \cdots (365 - n + 1)$$

Assuming that each outcome is equally likely, then by Eq. (*1.38*),

$$P(A) = \frac{n(A)}{n(S)} = \frac{(365)(364) \cdots (365 - n + 1)}{(365)^n} \tag{1.78}$$

Let $B$ be the event that at least two persons have the same birthday. Then $B = \bar{A}$ and by Eq. (*1.25*), $P(B) = 1 - P(A)$.

(*b*) Substituting $n = 50$ in Eq. (*1.78*), we have

$$P(A) \approx 0.03 \quad \text{and} \quad P(B) \approx 1 - 0.03 = 0.97$$

(*c*) From Eq. (*1.78*), when $n = 23$, we have

$$P(A) \approx 0.493 \quad \text{and} \quad P(B) = 1 - P(A) \approx 0.507$$

That is, if there are 23 persons in a room, the probability that at least two of them have the same birthday exceeds 0.5.

**1.33.** A committee of 5 persons is to be selected randomly from a group of 5 men and 10 women.

(*a*) Find the probability that the committee consists of 2 men and 3 women.

(*b*) Find the probability that the committee consists of all women.

(*a*) The number of total outcomes is given by

$$n(S) = \binom{15}{5}$$

It is assumed that "random selection" means that each of the outcomes is equally likely. Let $A$ be the event that the committee consists of 2 men and 3 women. Then the number of outcomes belonging to $A$ is given by

$$n(A) = \binom{5}{2}\binom{10}{3}$$

Thus, by Eq. (*1.38*),

$$P(A) = \frac{n(A)}{n(S)} = \frac{\binom{5}{2}\binom{10}{3}}{\binom{15}{5}} = \frac{400}{1001} \approx 0.4$$

(b) Let $B$ be the event that the committee consists of all women. Then the number of outcomes belonging to $B$ is

$$n(B) = \binom{5}{0}\binom{10}{5}$$

Thus, by Eq. (*1.38*),

$$P(B) = \frac{n(B)}{n(S)} = \frac{\binom{5}{0}\binom{10}{5}}{\binom{15}{5}} = \frac{36}{429} \approx 0.084$$

**1.34.** Consider the switching network shown in Fig. 1-12. It is equally likely that a switch will or will not work. Find the probability that a closed path will exist between terminals $a$ and $b$.

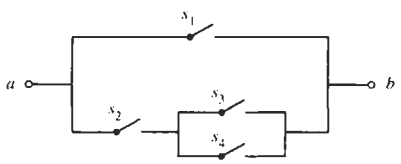



**Fig. 1-12**

Consider a sample space $S$ of which a typical outcome is (1, 0, 0, 1), indicating that switches 1 and 4 are closed and switches 2 and 3 are open. The sample space contains $2^4 = 16$ points, and by assumption, they are equally likely (Fig. 1-13).

Let $A_i$, $i = 1, 2, 3, 4$ be the event that the switch $s_i$ is closed. Let $A$ be the event that there exists a closed path between $a$ and $b$. Then

$$A = A_1 \cup (A_2 \cap A_3) \cup (A_2 \cap A_4)$$

Applying Eq. (*1.30*), we have

$$\begin{aligned} P(A) &= P[A_1 \cup (A_2 \cap A_3) \cup (A_2 \cap A_4)] \\ &= P(A_1) + P(A_2 \cap A_3) + P(A_2 \cap A_4) \\ &\quad - P[A_1 \cap (A_2 \cap A_3)] - P[A_1 \cap (A_2 \cap A_4)] - P[(A_2 \cap A_3) \cap (A_2 \cap A_4)] \\ &\quad + P[A_1 \cap (A_2 \cap A_3) \cap (A_2 \cap A_4)] \\ &= P(A_1) + P(A_2 \cap A_3) + P(A_2 \cap A_4) \\ &\quad - P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_4) - P(A_2 \cap A_3 \cap A_4) \\ &\quad + P(A_1 \cap A_2 \cap A_3 \cap A_4) \end{aligned}$$

Now, for example, the event $A_2 \cap A_3$ contains all elementary events with a 1 in the second and third places. Thus, from Fig. 1-13, we see that

$$\begin{aligned} &n(A_1) = 8 \qquad n(A_2 \cap A_3) = 4 \qquad n(A_2 \cap A_4) = 4 \\ &n(A_1 \cap A_2 \cap A_3) = 2 \qquad n(A_1 \cap A_2 \cap A_4) = 2 \\ &n(A_2 \cap A_3 \cap A_4) = 2 \qquad n(A_1 \cap A_2 \cap A_3 \cap A_4) = 1 \end{aligned}$$

Thus,

$$P(A) = \tfrac{8}{16} + \tfrac{4}{16} + \tfrac{4}{16} - \tfrac{2}{16} - \tfrac{2}{16} - \tfrac{2}{16} + \tfrac{1}{16} = \tfrac{11}{16} \approx 0.688$$

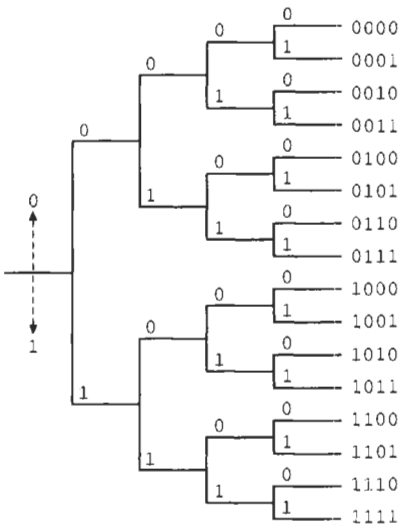



**Fig. 1-13**

**1.35.** Consider the experiment of tossing a fair coin repeatedly and counting the number of tosses required until the first head appears.

(*a*) Find the sample space of the experiment.

(*b*) Find the probability that the first head appears on the $k$th toss.

(*c*) Verify that $P(S) = 1$.

(*a*) The sample space of this experiment is

$$S = \{e_1, e_2, e_3, \ldots\} = \{e_k: k = 1, 2, 3, \ldots\}$$

where $e_k$ is the elementary event that the first head appears on the $k$th toss.

(*b*) Since a fair coin is tossed, we assume that a head and a tail are equally likely to appear. Then $P(H) = P(T) = \frac{1}{2}$. Let

$$P(e_k) = p_k \qquad k = 1, 2, 3, \ldots$$

Since there are $2^k$ equally likely ways of tossing a fair coin $k$ times, only one of which consists of $(k-1)$ tails following a head we observe that

$$P(e_k) = p_k = \frac{1}{2^k} \qquad k = 1, 2, 3, \ldots \tag{1.79}$$

(*c*) Using the power series summation formula, we have

$$P(S) = \sum_{k=1}^{\infty} P(e_k) = \sum_{k=1}^{\infty} \frac{1}{2^k} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \tag{1.80}$$

**1.36.** Consider the experiment of Prob. 1.35.

(*a*) Find the probability that the first head appears on an even-numbered toss.

(*b*) Find the probability that the first head appears on an odd-numbered toss.

(*a*) Let $A$ be the event "the first head appears on an even-numbered toss." Then, by Eq. (*1.36*) and using Eq. (*1.79*) of Prob. 1.35, we have

$$P(A) = p_2 + p_4 + p_6 + \cdots = \sum_{m=1}^{\infty} p_{2m} = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} = \sum_{m=1}^{\infty} \left(\frac{1}{4}\right)^m = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}$$

(*b*) Let $B$ be the event "the first head appears on an odd-numbered toss." Then it is obvious that $B = \bar{A}$. Then, by Eq. (*1.25*), we get

$$P(B) = P(\bar{A}) = 1 - P(A) = 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

As a check, notice that

$$P(B) = p_1 + p_3 + p_5 + \cdots = \sum_{m=0}^{\infty} p_{2m+1} = \sum_{m=0}^{\infty} \frac{1}{2^{2m+1}} = \frac{1}{2} \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m = \frac{1}{2}\left(\frac{1}{1 - \frac{1}{4}}\right) = \frac{2}{3}$$

## CONDITIONAL PROBABILITY

**1.37.** Show that $P(A \mid B)$ defined by Eq. (*1.39*) satisfies the three axions of a probability, that is,

(*a*) $P(A \mid B) \geq 0$

(*b*) $P(S \mid B) = 1$

(*c*) $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B)$ if $A_1 \cap A_2 = \varnothing$

(*a*) From definition (*1.39*),

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \qquad P(B) > 0$$

By axiom 1, $P(A \cap B) \geq 0$. Thus,

$$P(A \mid B) \geq 0$$

(*b*) By Eq. (1.5), $S \cap B = B$. Then

$$P(S \mid B) = \frac{P(S \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

(*c*) By definition (*1.39*),

$$P(A_1 \cup A_2 \mid B) = \frac{P[(A_1 \cup A_2) \cap B]}{P(B)}$$

Now by Eqs. (*1.8*) and (*1.11*), we have

$$(A_1 \cup A_2) \cap B = (A_1 \cap B) \cup (A_2 \cap B)$$

and $A_1 \cap A_2 = \varnothing$ implies that $(A_1 \cap B) \cap (A_2 \cap B) = \varnothing$. Thus, by axiom 3 we get

$$P(A_1 \cup A_2 \mid B) = \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$
$$= P(A_1 \mid B) + P(A_2 \mid B) \qquad \text{if } A_1 \cap A_2 = \varnothing$$

**1.38.** Find $P(A \mid B)$ if (*a*) $A \cap B = \varnothing$, (*b*) $A \subset B$, and (*c*) $B \subset A$.

(*a*) If $A \cap B = \varnothing$, then $P(A \cap B) = P(\varnothing) = 0$. Thus,

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\varnothing)}{P(B)} = 0$$

(*b*) If $A \subset B$, then $A \cap B = A$ and

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$$

(*c*) If $B \subset A$, then $A \cap B = B$ and

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$$

**1.39.** Show that if $P(A \mid B) > P(A)$, then $P(B \mid A) > P(B)$.

If $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} > P(A)$, then $P(A \cap B) > P(A)P(B)$. Thus,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} > \frac{P(A)P(B)}{P(A)} = P(B) \qquad \text{or} \qquad P(B \mid A) > P(B)$$

**1.40.** Consider the experiment of throwing the two fair dice of Prob. 1.31 behind you; you are then informed that the sum is not greater than 3.

(*a*) Find the probability of the event that two faces are the same without the information given.

(*b*) Find the probability of the same event with the information given.

(*a*) Let $A$ be the event that two faces are the same. Then from Fig. 1-3 (Prob. 1.5) and by Eq. (*1.38*), we have

$$A = \{(i, i): i = 1, 2, \ldots, 6\}$$

and

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

(*b*) Let $B$ be the event that the sum is not greater than 3. Again from Fig. 1-3, we see that

$$B = \{(i, j): i + j \leq 3\} = \{(1, 1), (1, 2), (2, 1)\}$$

and

$$P(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

Now $A \cap B$ is the event that two faces are the same and also that their sum is not greater than 3. Thus,

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

Then by definition (*1.39*), we obtain

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{1}{12}} = \frac{1}{3}$$

Note that the probability of the event that two faces are the same doubled from $\frac{1}{6}$ to $\frac{1}{3}$ with the information given.

*Alternative Solution:*

There are 3 elements in $B$, and 1 of them belongs to $A$. Thus, the probability of the same event with the information given is $\frac{1}{3}$.

**1.41.** Two manufacturing plants produce similar parts. Plant 1 produces 1,000 parts, 100 of which are defective. Plant 2 produces 2,000 parts, 150 of which are defective. A part is selected at random and found to be defective. What is the probability that it came from plant 1?

Let $B$ be the event that "the part selected is defective," and let $A$ be the event that "the part selected came from plant 1." Then $A \cap B$ is the event that the item selected is defective and came from plant 1. Since a part is selected at random, we assume equally likely events, and using Eq. (*1.38*), we have

$$P(A \cap B) = \frac{100}{3000} = \frac{1}{30}$$

Similarly, since there are 3000 parts and 250 of them are defective, we have

$$P(B) = \frac{250}{3000} = \frac{1}{12}$$

By Eq. (*1.39*), the probability that the part came from plant 1 is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{30}}{\frac{1}{12}} = \frac{2}{5} = 0.4$$

*Alternative Solution:*

There are 250 defective parts, and 100 of these are from plant 1. Thus, the probability that the defective part came from plant 1 is $\frac{100}{250} = 0.4$.

**1.42.** A lot of 100 semiconductor chips contains 20 that are defective. Two chips are selected at random, without replacement, from the lot.

(*a*) What is the probability that the first one selected is defective?

(*b*) What is the probability that the second one selected is defective given that the first one was defective?

(*c*) What is the probability that both are defective?

(*a*) Let $A$ denote the event that the first one selected is defective. Then, by Eq. (*1.38*),

$$P(A) = \tfrac{20}{100} = 0.2$$

(*b*) Let $B$ denote the event that the second one selected is defective. After the first one selected is defective, there are 99 chips left in the lot with 19 chips that are defective. Thus, the probability that the second one selected is defective given that the first one was defective is

$$P(B \mid A) = \tfrac{19}{99} = 0.192$$

(*c*) By Eq. (*1.41*), the probability that both are defective is

$$P(A \cap B) = P(B \mid A)P(A) = (\tfrac{19}{99})(0.2) = 0.0384$$

**1.43.** A number is selected at random from $\{1, 2, \ldots, 100\}$. Given that the number selected is divisible by 2, find the probability that it is divisible by 3 or 5.

Let
$A_2$ = event that the number is divisible by 2
$A_3$ = event that the number is divisible by 3
$A_5$ = event that the number is divisible by 5

Then the desired probability is

$$P(A_3 \cup A_5 \mid A_2) = \frac{P[(A_3 \cup A_5) \cap A_2]}{P(A_2)} \quad \text{[Eq. (1.39)]}$$

$$= \frac{P[(A_3 \cap A_2) \cup (A_5 \cap A_2)]}{P(A_2)} \quad \text{[Eq. (1.12)]}$$

$$= \frac{P(A_3 \cap A_2) + P(A_5 \cap A_2) - P(A_3 \cap A_5 \cap A_2)}{P(A_2)} \quad \text{[Eq. (1.29)]}$$

Now
$A_3 \cap A_2$ = event that the number is divisible by 6
$A_5 \cap A_2$ = event that the number is divisible by 10
$A_3 \cap A_5 \cap A_2$ = event that the number is divisible by 30

and $P(A_3 \cap A_2) = \tfrac{16}{100}$ $P(A_5 \cap A_2) = \tfrac{10}{100}$ $P(A_3 \cap A_5 \cap A_2) = \tfrac{3}{100}$

Thus, $$P(A_3 \cup A_5 \mid A_2) = \frac{\frac{16}{100} + \frac{10}{100} - \frac{3}{100}}{\frac{50}{100}} = \frac{23}{50} = 0.46$$

**1.44.** Let $A_1, A_2, \ldots, A_n$ be events in a sample space $S$. Show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap A_2 \cap \cdots \cap A_{n-1}) \tag{1.81}$$

We prove Eq. (*1.81*) by induction. Suppose Eq. (*1.81*) is true for $n = k$:

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_k \mid A_1 \cap A_2 \cap \cdots \cap A_{k-1})$$

Multiplying both sides by $P(A_{k+1} \mid A_1 \cap A_2 \cap \cdots \cap A_k)$, we have

$$P(A_1 \cap A_2 \cap \cdots \cap A_k)P(A_{k+1} \mid A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1 \cap A_2 \cap \cdots \cap A_{k+1})$$

and $$P(A_1 \cap A_2 \cap \cdots \cap A_{k+1}) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \cdots P(A_{k+1} \mid A_1 \cap A_2 \cap \cdots \cap A_k)$$

Thus, Eq. (*1.81*) is also true for $n = k + 1$. By Eq. (*1.41*), Eq. (*1.81*) is true for $n = 2$. Thus Eq. (*1.81*) is true for $n \geq 2$.

**1.45.** Two cards are drawn at random from a deck. Find the probability that both are aces.

Let $A$ be the event that the first card is an ace, and $B$ be the event that the second card is an ace. The desired probability is $P(B \cap A)$. Since a card is drawn at random, $P(A) = \tfrac{4}{52}$. Now if the first card is an ace, then there will be 3 aces left in the deck of 51 cards. Thus $P(B \mid A) = \tfrac{3}{51}$. By Eq. (*1.41*),

$$P(B \cap A) = P(B \mid A)P(A) = (\tfrac{3}{51})(\tfrac{4}{52}) = \tfrac{1}{221}$$

Check:

By counting technique, we have

$$P(B \cap A) = \frac{\binom{4}{2}}{\binom{52}{2}} = \frac{(4)(3)}{(52)(51)} = \frac{1}{221}$$

**1.46.** There are two identical decks of cards, each possessing a distinct symbol so that the cards from each deck can be identified. One deck of cards is laid out in a fixed order, and the other deck is shuffled and the cards laid out one by one on top of the fixed deck. Whenever two cards with the same symbol occur in the same position, we say that a *match* has occurred. Let the number of cards in the deck be 10. Find the probability of getting a match at the first four positions.

Let $A_i$, $i = 1, 2, 3, 4$, be the events that a match occurs at the $i$th position. The required probability is

$$P(A_1 \cap A_2 \cap A_3 \cap A_4)$$

By Eq. (*1.81*),

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2)P(A_4 \mid A_1 \cap A_2 \cap A_3)$$

There are 10 cards that can go into position 1, only one of which matches. Thus, $P(A_1) = \frac{1}{10}$. $P(A_2 \mid A_1)$ is the conditional probability of a match at position 2 given a match at position 1. Now there are 9 cards left to go into position 2, only one of which matches. Thus, $P(A_2 \mid A_1) = \frac{1}{9}$. In a similar fashion, we obtain $P(A_3 \mid A_1 \cap A_2) = \frac{1}{8}$ and $P(A_4 \mid A_1 \cap A_2 \cap A_3) = \frac{1}{7}$. Thus,

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = (\tfrac{1}{10})(\tfrac{1}{9})(\tfrac{1}{8})(\tfrac{1}{7}) = \tfrac{1}{3040}$$

## TOTAL PROBABILITY

**1.47.** Verify Eq. (*1.44*).

Since $B \cap S = B$ [and using Eq. (*1.43*)], we have

$$\begin{aligned} B = B \cap S &= B \cap (A_1 \cup A_2 \cup \cdots \cup A_n) \\ &= (B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_n) \end{aligned} \tag{1.82}$$

Now the events $B \cap A_i$, $i = 1, 2, \ldots, n$, are mutually exclusive, as seen from the Venn diagram of Fig. 1-14. Then by axiom 3 of probability and Eq. (*1.41*), we obtain

$$P(B) = P(B \cap S) = \sum_{i=1}^{n} P(B \cap A_i) = \sum_{i=1}^{n} P(B \mid A_i)P(A_i)$$

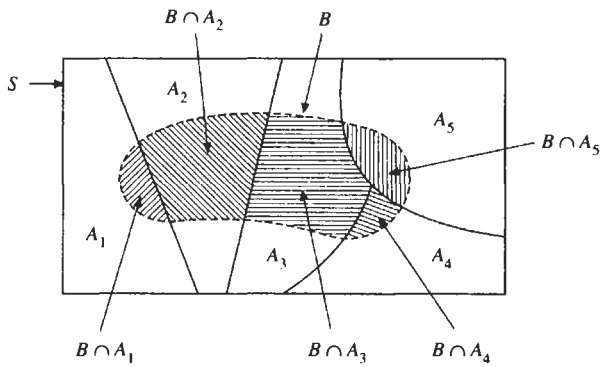



Fig. 1-14

**1.48.** Show that for any events $A$ and $B$ in $S$,

$$P(B) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A}) \tag{1.83}$$

From Eq. (*1.64*) (Prob. 1.23), we have

$$P(B) = P(B \cap A) + P(B \cap \bar{A})$$

Using Eq. (*1.39*), we obtain

$$P(B) = P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A})$$

Note that Eq. (*1.83*) is the special case of Eq. (*1.44*).

**1.49.** Suppose that a laboratory test to detect a certain disease has the following statistics. Let

$A$ = event that the tested person has the disease

$B$ = event that the test result is positive

It is known that

$$P(B \mid A) = 0.99 \qquad \text{and} \qquad P(B \mid \bar{A}) = 0.005$$

and 0.1 percent of the population actually has the disease. What is the probability that a person has the disease given that the test result is positive?

From the given statistics, we have

$$P(A) = 0.001 \qquad \text{then} \qquad P(\bar{A}) = 0.999$$

The desired probability is $P(A \mid B)$. Thus, using Eqs. (*1.42*) and (*1.83*), we obtain

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid \bar{A})P(\bar{A})}$$
$$= \frac{(0.99)(0.001)}{(0.99)(0.001) + (0.005)(0.999)} = 0.165$$

Note that in only 16.5 percent of the cases where the tests are positive will the person actually have the disease even though the test is 99 percent effective in detecting the disease when it is, in fact, present.

**1.50.** A company producing electric relays has three manufacturing plants producing 50, 30, and 20 percent, respectively, of its product. Suppose that the probabilities that a relay manufactured by these plants is defective are 0.02, 0.05, and 0.01, respectively.

(*a*) If a relay is selected at random from the output of the company, what is the probability that it is defective?

(*b*) If a relay selected at random is found to be defective, what is the probability that it was manufactured by plant 2?

(*a*) Let $B$ be the event that the relay is defective, and let $A_i$ be the event that the relay is manufactured by plant $i$ ($i = 1, 2, 3$). The desired probability is $P(B)$. Using Eq. (*1.44*), we have

$$P(B) = \sum_{i=1}^{3} P(B \mid A_i)P(A_i)$$
$$= (0.02)(0.5) + (0.05)(0.3) + (0.01)(0.2) = 0.027$$

(*b*) The desired probability is $P(A_2 \mid B)$. Using Eq. (1.42) and the result from part (*a*), we obtain

$$P(A_2 \mid B) = \frac{P(B \mid A_2)P(A_2)}{P(B)} = \frac{(0.05)(0.3)}{0.027} = 0.556$$

**1.51.** Two numbers are chosen at random from among the numbers 1 to 10 without replacement. Find the probability that the second number chosen is 5.

Let $A_i$, $i = 1, 2, \ldots, 10$ denote the event that the first number chosen is $i$. Let $B$ be the event that the second number chosen is 5. Then by Eq. (*1.44*),

$$P(B) = \sum_{i=1}^{10} P(B \mid A_i)P(A_i)$$

Now $P(A_i) = \frac{1}{10}$. $P(B \mid A_i)$ is the probability that the second number chosen is 5, given that the first is $i$. If $i = 5$, then $P(B \mid A_i) = 0$. If $i \neq 5$, then $P(B \mid A_i) = \frac{1}{9}$. Hence,

$$P(B) = \sum_{i=1}^{10} P(B \mid A_i)P(A_i) = 9(\tfrac{1}{9})(\tfrac{1}{10}) = \tfrac{1}{10}$$

**1.52.** Consider the binary communication channel shown in Fig. 1-15. The channel input symbol $X$ may assume the state 0 or the state 1, and, similarly, the channel output symbol $Y$ may assume either the state 0 or the state 1. Because of the channel noise, an input 0 may convert to an output 1 and vice versa. The channel is characterized by the channel transition probabilities $p_0$, $q_0$, $p_1$, and $q_1$, defined by

$$p_0 = P(y_1 \mid x_0) \quad \text{and} \quad p_1 = P(y_0 \mid x_1)$$
$$q_0 = P(y_0 \mid x_0) \quad \text{and} \quad q_1 = P(y_1 \mid x_1)$$

where $x_0$ and $x_1$ denote the events $(X = 0)$ and $(X = 1)$, respectively, and $y_0$ and $y_1$ denote the events $(Y = 0)$ and $(Y = 1)$, respectively. Note that $p_0 + q_0 = 1 = p_1 + q_1$. Let $P(x_0) = 0.5$, $p_0 = 0.1$, and $p_1 = 0.2$.

(*a*) Find $P(y_0)$ and $P(y_1)$.

(*b*) If a 0 was observed at the output, what is the probability that a 0 was the input state?

(*c*) If a 1 was observed at the output, what is the probability that a 1 was the input state?

(*d*) Calculate the probability of error $P_e$.

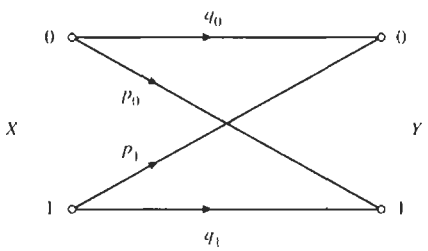



**Fig. 1-15**

(*a*) We note that

$$P(x_1) = 1 - P(x_0) = 1 - 0.5 = 0.5$$
$$P(y_0 \mid x_0) = q_0 = 1 - p_0 = 1 - 0.1 = 0.9$$
$$P(y_1 \mid x_1) = q_1 = 1 - p_1 = 1 - 0.2 = 0.8$$

Using Eq. (*1.44*), we obtain

$$P(y_0) = P(y_0 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.9(0.5) + 0.2(0.5) = 0.55$$
$$P(y_1) = P(y_1 \mid x_0)P(x_0) + P(y_1 \mid x_1)P(x_1) = 0.1(0.5) + 0.8(0.5) = 0.45$$

(*b*) Using Bayes' rule (*1.42*), we have

$$P(x_0 \mid y_0) = \frac{P(x_0)P(y_0 \mid x_0)}{P(y_0)} = \frac{(0.5)(0.9)}{0.55} = 0.818$$

(*c*) Similarly,

$$P(x_1 \mid y_1) = \frac{P(x_1)P(y_1 \mid x_1)}{P(y_1)} = \frac{(0.5)(0.8)}{0.45} = 0.889$$

(*d*) The probability of error is

$$P_e = P(y_1 \mid x_0)P(x_0) + P(y_0 \mid x_1)P(x_1) = 0.1(0.5) + 0.2(0.5) = 0.15.$$

## INDEPENDENT EVENTS

**1.53.** Let $A$ and $B$ be events in a sample space $S$. Show that if $A$ and $B$ are independent, then so are (*a*) $A$ and $\bar{B}$, (*b*) $\bar{A}$ and $B$, and (*c*) $\bar{A}$ and $\bar{B}$.

(*a*) From Eq. (*1.64*) (Prob. 1.23), we have

$$P(A) = P(A \cap B) + P(A \cap \bar{B})$$

Since $A$ and $B$ are independent, using Eqs. (*1.46*) and (*1.25*), we obtain

$$\begin{aligned} P(A \cap \bar{B}) &= P(A) - P(A \cap B) = P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] = P(A)P(\bar{B}) \end{aligned} \tag{1.84}$$

Thus, by definition (*1.46*), $A$ and $\bar{B}$ are independent.

(b) Interchanging $A$ and $B$ in Eq. (*1.84*), we obtain

$$P(B \cap \bar{A}) = P(B)P(\bar{A})$$

which indicates that $\bar{A}$ and $B$ are independent.

(*c*) We have

$$\begin{aligned} P(\bar{A} \cap \bar{B}) &= P[\overline{(A \cup B)}] && \text{[Eq. (1.14)]} \\ &= 1 - P(A \cup B) && \text{[Eq. (1.25)]} \\ &= 1 - P(A) - P(B) + P(A \cap B) && \text{[Eq. (1.29)]} \\ &= 1 - P(A) - P(B) + P(A)P(B) && \text{[Eq. (1.46)]} \\ &= 1 - P(A) - P(B)[1 - P(A)] \\ &= [1 - P(A)][1 - P(B)] \\ &= P(\bar{A})P(\bar{B}) && \text{[Eq. (1.25)]} \end{aligned}$$

Hence, $\bar{A}$ and $\bar{B}$ are independent.

**1.54.** Let $A$ and $B$ be events defined in a sample space $S$. Show that if both $P(A)$ and $P(B)$ are nonzero, then events $A$ and $B$ cannot be both mutually exclusive and independent.

Let $A$ and $B$ be mutually exclusive events and $P(A) \neq 0$, $P(B) \neq 0$. Then $P(A \cap B) = P(\varnothing) = 0$ but $P(A)P(B) \neq 0$. Since

$$P(A \cap B) \neq P(A)P(B)$$

$A$ and $B$ cannot be independent.

**1.55.** Show that if three events $A$, $B$, and $C$ are independent, then $A$ and $(B \cup C)$ are independent.

We have

$$\begin{aligned} P[A \cap (B \cup C)] &= P[(A \cap B) \cup (A \cap C)] && \text{[Eq. (1.12)]} \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) && \text{[Eq. (1.29)]} \\ &= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C) && \text{[Eq. (1.50)]} \\ &= P(A)P(B) + P(A)P(C) - P(A)P(B \cap C) && \text{[Eq. (1.50)]} \\ &= P(A)[P(B) + P(C) - P(B \cap C)] \\ &= P(A)P(B \cup C) && \text{[Eq. (1.29)]} \end{aligned}$$

Thus, $A$ and $(B \cup C)$ are independent.

**1.56.** Consider the experiment of throwing two fair dice (Prob. 1.31). Let $A$ be the event that the sum of the dice is 7, $B$ be the event that the sum of the dice is 6, and $C$ be the event that the first die is 4. Show that events $A$ and $C$ are independent, but events $B$ and $C$ are not independent.

From Fig. 1-3 (Prob. 1.5), we see that

$$A = \{\zeta_{16}, \zeta_{25}, \zeta_{34}, \zeta_{43}, \zeta_{52}, \zeta_{61}\}$$
$$B = \{\zeta_{15}, \zeta_{24}, \zeta_{33}, \zeta_{42}, \zeta_{51}\}$$
$$C = \{\zeta_{41}, \zeta_{42}, \zeta_{43}, \zeta_{44}, \zeta_{45}, \zeta_{46}\}$$

and $$A \cap C = \{\zeta_{43}\} \qquad B \cap C = \{\zeta_{42}\}$$

Now $$P(A) = \tfrac{6}{36} = \tfrac{1}{6} \qquad P(B) = \tfrac{5}{36} \qquad P(C) = \tfrac{6}{36} = \tfrac{1}{6}$$

and $$P(A \cap C) = \tfrac{1}{36} = P(A)P(C)$$

Thus, events $A$ and $C$ are independent. But

$$P(B \cap C) = \tfrac{1}{36} \neq P(B)P(C)$$

Thus, events $B$ and $C$ are not independent.

**1.57.** In the experiment of throwing two fair dice, let $A$ be the event that the first die is odd, $B$ be the event that the second die is odd, and $C$ be the event that the sum is odd. Show that events $A$, $B$, and $C$ are pairwise independent, but $A$, $B$, and $C$ are not independent.

From Fig. 1-3 (Prob. 1.5), we see that

$$P(A) = P(B) = P(C) = \tfrac{18}{36} = \tfrac{1}{2}$$
$$P(A \cap B) = P(A \cap C) = P(B \cap C) = \tfrac{9}{36} = \tfrac{1}{4}$$

Thus
$$P(A \cap B) = \tfrac{1}{4} = P(A)P(B)$$
$$P(A \cap C) = \tfrac{1}{4} = P(A)P(C)$$
$$P(B \cap C) = \tfrac{1}{4} = P(B)P(C)$$

which indicates that $A$, $B$, and $C$ are pairwise independent. However, since the sum of two odd numbers is even, $\{A \cap B \cap C) = \varnothing$ and

$$P(A \cap B \cap C) = 0 \neq \tfrac{1}{8} = P(A)P(B)P(C)$$

which shows that $A$, $B$, and $C$ are not independent.

**1.58.** A system consisting of $n$ separate components is said to be a *series* system if it functions when all $n$ components function (Fig. 1-16). Assume that the components fail independently and that the probability of failure of component $i$ is $p_i$, $i = 1, 2, \ldots, n$. Find the probability that the system functions.

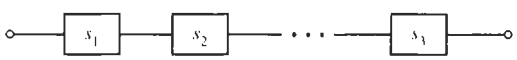



**Fig. 1-16** Series system.

Let $A_i$ be the event that component $s_i$ functions. Then

$$P(A_i) = 1 - P(\bar{A}_i) = 1 - p_i$$

Let $A$ be the event that the system functions. Then, since $A_i$'s are independent, we obtain

$$P(A) = P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i) = \prod_{i=1}^{n} (1 - p_i) \tag{1.85}$$

**1.59.** A system consisting of $n$ separate components is said to be a *parallel* system if it functions when at least one of the components functions (Fig. 1-17). Assume that the components fail independently and that the probability of failure of component $i$ is $p_i$, $i = 1, 2, \ldots, n$. Find the probability that the system functions.

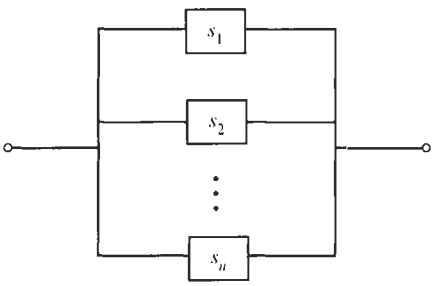



**Fig. 1-17** Parallel system.

Let $A_i$ be the event that component $s_i$ functions. Then

$$P(\bar{A}_i) = p_i$$

Let $A$ be the event that the system functions. Then, since $\bar{A}_i$'s are independent, we obtain

$$P(A) = 1 - P(\bar{A}) = 1 - P\left(\bigcap_{i=1}^{n} \bar{A}_i\right) = 1 - \prod_{i=1}^{n} p_i \tag{1.86}$$

**1.60.** Using Eqs. (*1.85*) and (*1.86*), redo Prob. 1.34.

From Prob. 1.34, $p_i = \frac{1}{2}$, $i = 1, 2, 3, 4$, where $p_i$ is the probability of failure of switch $s_i$. Let $A$ be the event that there exists a closed path between $a$ and $b$. Using Eq. (*1.86*), the probability of failure for the parallel combination of switches 3 and 4 is

$$p_{34} = p_3 p_4 = (\tfrac{1}{2})(\tfrac{1}{2}) = \tfrac{1}{4}$$

Using Eq. (*1.85*), the probability of failure for the combination of switches 2, 3, and 4 is

$$p_{234} = 1 - (1 - \tfrac{1}{2})(1 - \tfrac{1}{4}) = 1 - \tfrac{3}{8} = \tfrac{5}{8}$$

Again, using Eq. (*1.86*), we obtain

$$P(A) = 1 - p_1 p_{234} = 1 - (\tfrac{1}{2})(\tfrac{5}{8}) = 1 - \tfrac{5}{16} = \tfrac{11}{16}$$

**1.61.** A *Bernoulli* experiment is a random experiment, the outcome of which can be classified in but one of two mutually exclusive and exhaustive ways, say success or failure. A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several independent times so that the probability of success, say $p$, remains the same from trial to trial. Now an infinite sequence of Bernoulli trials is performed. Find the probability that (*a*) at least 1 success occurs in the first $n$ trials; (*b*) exactly $k$ successes occur in the first $n$ trials; (*c*) all trials result in successes.

(*a*) In order to find the probability of at least 1 success in the first $n$ trials, it is easier to first compute the probability of the complementary event, that of no successes in the first $n$ trials. Let $A_i$ denote the event

of a failure on the $i$th trial. Then the probability of no successes is, by independence,

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = P(A_1)P(A_2) \cdots P(A_n) = (1-p)^n \tag{1.87}$$

Hence, the probability that at least 1 success occurs in the first $n$ trials is $1 - (1-p)^n$.

(*b*) In any particular sequence of the first $n$ outcomes, if $k$ successes occur, where $k = 0, 1, 2, \ldots, n$, then $n - k$ failures occur. There are $\binom{n}{k}$ such sequences, and each one of these has probability $p^k(1-p)^{n-k}$. Thus, the probability that exactly $k$ successes occur in the first $n$ trials is given by $\binom{n}{k}p^k(1-p)^{n-k}$.

(*c*) Since $\bar{A}_i$ denotes the event of a success on the $i$th trial, the probability that all trials resulted in successes in the first $n$ trials is, by independence,

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n) = P(\bar{A}_1)P(\bar{A}_2) \cdots P(\bar{A}_n) = p^n \tag{1.88}$$

Hence, using the continuity theorem of probability (*1.74*) (Prob. 1.28), the probability that all trials result in successes is given by

$$P\left(\bigcap_{i=1}^{\infty} \bar{A}_i\right) = P\left(\lim_{n\to\infty} \bigcap_{i=1}^{n} \bar{A}_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} \bar{A}_i\right) = \lim_{n\to\infty} p^n = \begin{cases} 0 & p < 1 \\ 1 & p = 1 \end{cases}$$

**1.62.** Let $S$ be the sample space of an experiment and $S = \{A, B, C\}$, where $P(A) = p$, $P(B) = q$, and $P(C) = r$. The experiment is repeated infinitely, and it is assumed that the successive experiments are independent. Find the probability of the event that $A$ occurs before $B$.

Suppose that $A$ occurs for the first time at the $n$th trial of the experiment. If $A$ is to have occurred before $B$, then $C$ must have occurred on the first $(n-1)$ trials. Let $D$ be the event that $A$ occurs before $B$. Then

$$D = \bigcup_{n=1}^{\infty} D_n$$

where $D_n$ is the event that $C$ occurs on the first $(n-1)$ trials and $A$ occurs on the $n$th trial. Since $D_n$'s are mutually exclusive, we have

$$P(D) = \sum_{n=1}^{\infty} P(D_n)$$

Since the trials are independent, we have

$$P(D_n) = [P(C)]^{n-1}P(A) = r^{n-1}p$$

Thus,

$$P(D) = \sum_{n=1}^{\infty} r^{n-1}p = p\sum_{k=0}^{\infty} r^k = \frac{p}{1-r} = \frac{p}{p+q}$$

or

$$P(D) = \frac{P(A)}{P(A) + P(B)} \tag{1.89}$$

since $p + q + r = 1$.

**1.63.** In a gambling game, craps, a pair of dice is rolled and the outcome of the experiment is the sum of the dice. The player wins on the first roll if the sum is 7 or 11 and loses if the sum is 2, 3, or 12. If the sum is 4, 5, 6, 8, 9, or 10, that number is called the player's "point." Once the point is established, the rule is: If the player rolls a 7 before the point, the player loses; but if the point is rolled before a 7, the player wins. Compute the probability of winning in the game of craps.

Let $A$, $B$, and $C$ be the events that the player wins, the player wins on the first roll, and the player gains point, respectively. Then $P(A) = P(B) + P(C)$. Now from Fig. 1-3 (Prob. 1.5),

$$P(B) = P(\text{sum} = 7) + P(\text{sum} = 11) = \tfrac{6}{36} + \tfrac{2}{36} = \tfrac{2}{9}$$

Let $A_k$ be the event that point of $k$ occurs before 7. Then

$$P(C) = \sum_{k \in \{4,\ 5,\ 6,\ 8,\ 9,\ 10\}} P(A_k)P(\text{point} = k)$$

By Eq. (*1.89*) (Prob. 1.62),

$$P(A_k) = \frac{P(\text{sum} = k)}{P(\text{sum} = k) + P(\text{sum} = 7)} \tag{1.90}$$

Again from Fig. 1-3,

$$P(\text{sum} = 4) = \tfrac{3}{36} \qquad P(\text{sum} = 5) = \tfrac{4}{36} \qquad P(\text{sum} = 6) = \tfrac{5}{36}$$
$$P(\text{sum} = 8) = \tfrac{5}{36} \qquad P(\text{sum} = 9) = \tfrac{4}{36} \qquad P(\text{sum} = 10) = \tfrac{3}{36}$$

Now by Eq. (*1.90*),

$$P(A_4) = \tfrac{1}{3} \qquad P(A_5) = \tfrac{2}{5} \qquad P(A_6) = \tfrac{5}{11}$$
$$P(A_8) = \tfrac{5}{11} \qquad P(A_9) = \tfrac{2}{5} \qquad P(A_{10}) = \tfrac{1}{3}$$

Using these values, we obtain

$$P(A) = P(B) + P(C) = \tfrac{2}{9} + \tfrac{134}{495} = 0.49293$$

## Supplementary Problems

**1.64.** Consider the experiment of selecting items from a group consisting of three items $\{a, b, c\}$.

(*a*) Find the sample space $S_1$ of the experiment in which two items are selected without replacement.

(*b*) Find the sample space $S_2$ of the experiment in which two items are selected with replacement.

*Ans.* (*a*) $S_1 = \{ab, ac, ba, bc, ca, cb\}$

(*b*) $S_2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

**1.65.** Let $A$ and $B$ be arbitrary events. Then show that $A \subset B$ if and only if $A \cup B = B$.

*Hint:* Draw a Venn diagram.

**1.66.** Let $A$ and $B$ be events in the sample space $S$. Show that if $A \subset B$, then $\bar{B} \subset \bar{A}$.

*Hint:* Draw a Venn diagram.

**1.67.** Verify Eq. (*1.13*).

*Hint:* Draw a Venn diagram.

**1.68.** Let $A$ and $B$ be any two events in $S$. The *difference* of $B$ and $A$, denoted by $B - A$, is defined as

$$B - A = B \cap \bar{A}$$

The *symmetric difference* of $A$ and $B$, denoted by $A \triangle B$, is defined by

$$A \triangle B = (A - B) \cup (B - A)$$

Show that

$$A \triangle B = (A \cup B) \cap \overline{(A \cap B)}$$

*Hint:* Draw a Venn diagram.

**1.69.** Let $A$ and $B$ be any two events in $S$. Express the following events in terms of $A$ and $B$.

(*a*) At least one of the events occurs.

(*b*) Exactly one of two events occurs.

*Ans.* (*a*) $A \cup B$; (*b*) $A \triangle B$

**1.70.** Let $A$, $B$, and $C$ be any three events in $S$. Express the following events in terms of these events.

(*a*) Either $B$ or $C$ occurs, but not $A$.

(*b*) Exactly one of the events occurs.

(*c*) Exactly two of the events occur.

*Ans.* (*a*) $\bar{A} \cap (B \cup C)$

(*b*) $\{A \cap (\overline{B \cup C})\} \cup \{B \cap (\overline{A \cup C})\} \cup \{C \cap (\overline{A \cup B})\}$

(*c*) $\{(A \cap B) \cap \bar{C}\} \cup \{(A \cap C) \cap \bar{B}\} \cup \{(B \cap C) \cap \bar{A}\}$

**1.71.** A random experiment has sample space $S = \{a, b, c\}$. Suppose that $P(\{a, c\}) = 0.75$ and $P(\{b, c)\} = 0.6$. Find the probabilities of the elementary events.

*Ans.* $P(a) = 0.4$, $P(b) = 0.25$, $P(c) = 0.35$

**1.72.** Show that

(*a*) $P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B)$

(*b*) $P(A \cap B) \geq 1 - P(\bar{A}) - P(\bar{B})$

(*c*) $P(A \triangle B) = P(A \cup B) - P(A \cap B)$

*Hint:* (*a*) Use Eqs. (*1.15*) and (*1.25*).

(*b*) Use Eqs. (*1.29*), (*1.25*), and (*1.28*).

(*c*) See Prob. 1.68 and use axiom 3.

**1.73.** Let $A$, $B$, and $C$ be three events in $S$. If $P(A) = P(B) = \frac{1}{4}$, $P(C) = \frac{1}{3}$, $P(A \cap B) = \frac{1}{8}$, $P(A \cap C) = \frac{1}{6}$, and $P(B \cap C) = 0$, find $P(A \cup B \cup C)$.

*Ans.* $\frac{13}{24}$

**1.74.** Verify Eq. (*1.30*).

*Hint:* Prove by induction.

**1.75.** Show that

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n - 1)$$

*Hint:* Use induction to generalize Bonferroni's inequality (*1.63*) (Prob. 1.22).

**1.76.** In an experiment consisting of 10 throws of a pair of fair dice, find the probability of the event that at least one double 6 occurs.

*Ans.* 0.246

**1.77.** Show that if $P(A) > P(B)$, then $P(A \mid B) > P(B \mid A)$.

*Hint:* Use Eqs. (*1.39*) and (*1.40*).

**1.78.** An urn contains 8 white balls and 4 red balls. The experiment consists of drawing 2 balls from the urn without replacement. Find the probability that both balls drawn are white.

*Ans.* 0.424

**1.79.** There are 100 patients in a hospital with a certain disease. Of these, 10 are selected to undergo a drug treatment that increases the percentage cured rate from 50 percent to 75 percent. What is the probability that the patient received a drug treatment if the patient is known to be cured?

*Ans.* 0.143

**1.80.** Two boys and two girls enter a music hall and take four seats at random in a row. What is the probability that the girls take the two end seats?

*Ans.* $\frac{1}{6}$

**1.81.** Let $A$ and $B$ be two independent events in $S$. It is known that $P(A \cap B) = 0.16$ and $P(A \cup B) = 0.64$. Find $P(A)$ and $P(B)$.

*Ans.* $P(A) = P(B) = 0.4$

**1.82.** The relay network shown in Fig. 1-18 operates if and only if there is a closed path of relays from left to right. Assume that relays fail independently and that the probability of failure of each relay is as shown. What is the probability that the relay network operates?

*Ans.* 0.865

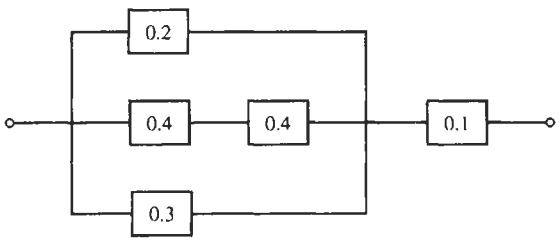



**Fig. 1-18**

# Chapter 2

# Random Variables

## 2.1 INTRODUCTION

In this chapter, the concept of a random variable is introduced. The main purpose of using a random variable is so that we can define certain probability functions that make it both convenient and easy to compute the probabilities of various events.

## 2.2 RANDOM VARIABLES

### A. Definitions:

Consider a random experiment with sample space $S$. A *random variable* $X(\zeta)$ is a single-valued real function that assigns a real number called the *value* of $X(\zeta)$ to each sample point $\zeta$ of $S$. Often, we use a single letter $X$ for this function in place of $X(\zeta)$ and use r.v. to denote the random variable.

Note that the terminology used here is traditional. Clearly a random variable is not a variable at all in the usual sense, and it is a function.

The sample space $S$ is termed the *domain* of the r.v. $X$, and the collection of all numbers [values of $X(\zeta)$] is termed the *range* of the r.v. $X$. Thus the range of $X$ is a certain subset of the set of all real numbers (Fig. 2-1).

Note that two or more different sample points might give the same value of $X(\zeta)$, but two different numbers in the range cannot be assigned to the same sample point.

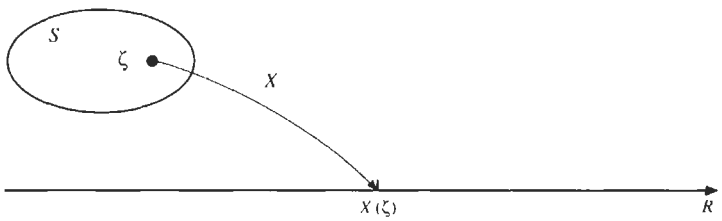



**Fig. 2-1** Random variable $X$ as a function.

**EXAMPLE 2.1** In the experiment of tossing a coin once (Example 1.1), we might define the r.v. $X$ as (Fig. 2-2)

$$X(H) = 1 \qquad X(T) = 0$$

Note that we could also define another r.v., say $Y$ or $Z$, with

$$Y(H) = 0,\ Y(T) = 1 \quad \text{or} \quad Z(H) = 0,\ Z(T) = 0$$

### B. Events Defined by Random Variables:

If $X$ is a r.v. and $x$ is a fixed real number, we can define the event $(X = x)$ as

$$(X = x) = \{\zeta\colon X(\zeta) = x\} \tag{2.1}$$

Similarly, for fixed numbers $x$, $x_1$, and $x_2$, we can define the following events:

$$\begin{aligned}(X \le x) &= \{\zeta\colon X(\zeta) \le x\}\\ (X > x) &= \{\zeta\colon X(\zeta) > x\}\\ (x_1 < X \le x_2) &= \{\zeta\colon x_1 < X(\zeta) \le x_2\}\end{aligned} \tag{2.2}$$

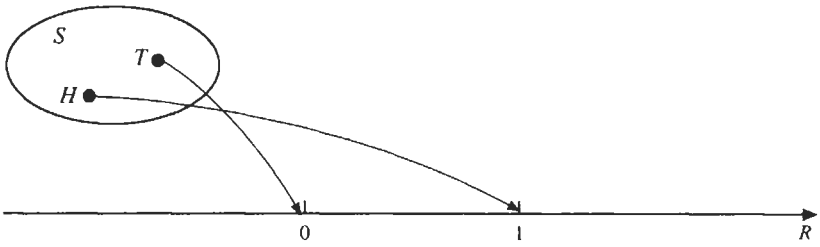



**Fig. 2-2** One random variable associated with coin tossing.

These events have probabilities that are denoted by

$$\begin{aligned} P(X = x) &= P\{\zeta\colon X(\zeta) = x\} \\ P(X \le x) &= P\{\zeta\colon X(\zeta) \le x\} \\ P(X > x) &= P\{\zeta\colon X(\zeta) > x\} \\ P(x_1 < X \le x_2) &= P\{\zeta\colon x_1 < X(\zeta) \le x_2\} \end{aligned} \tag{2.3}$$

**EXAMPLE 2.2** In the experiment of tossing a fair coin three times (Prob. 1.1), the sample space $S_1$ consists of eight equally likely sample points $S_1 = \{HHH, \ldots, TTT\}$. If $X$ is the r.v. giving the number of heads obtained, find (*a*) $P(X = 2)$; (*b*) $P(X < 2)$.

(*a*) Let $A \subset S_1$ be the event defined by $X = 2$. Then, from Prob. 1.1, we have

$$A = (X = 2) = \{\zeta\colon X(\zeta) = 2\} = \{HHT, HTH, THH\}$$

Since the sample points are equally likely, we have

$$P(X = 2) = P(A) = \tfrac{3}{8}$$

(*b*) Let $B \subset S_1$ be the event defined by $X < 2$. Then

$$B = (X < 2) = \{\zeta\colon X(\zeta) < 2\} = \{HTT, THT, TTH, TTT\}$$

and

$$P(X < 2) = P(B) = \tfrac{4}{8} = \tfrac{1}{2}$$

## 2.3 DISTRIBUTION FUNCTIONS

### A. Definition:

The *distribution function* [or *cumulative distribution function* (cdf)] of $X$ is the function defined by

$$F_X(x) = P(X \le x) \qquad -\infty < x < \infty \tag{2.4}$$

Most of the information about a random experiment described by the r.v. $X$ is determined by the behavior of $F_X(x)$.

### B. Properties of $F_X(x)$:

Several properties of $F_X(x)$ follow directly from its definition (2.4).

1. $0 \le F_X(x) \le 1$ (2.5)
2. $F_X(x_1) \le F_X(x_2)$ if $x_1 < x_2$ (2.6)
3. $\lim_{x \to \infty} F_X(x) = F_X(\infty) = 1$ (2.7)
4. $\lim_{x \to -\infty} F_X(x) = F_X(-\infty) = 0$ (2.8)
5. $\lim_{x \to a^+} F_X(x) = F_X(a^+) = F_X(a) \qquad a^+ = \lim_{0 < \varepsilon \to 0} a + \varepsilon$ (2.9)

Property 1 follows because $F_X(x)$ is a probability. Property 2 shows that $F_X(x)$ is a nondecreasing function (Prob. 2.5). Properties 3 and 4 follow from Eqs. (*1.22*) and (*1.26*):

$$\lim_{x \to \infty} P(X \leq x) = P(X \leq \infty) = P(S) = 1$$

$$\lim_{x \to -\infty} P(X \leq x) = P(X \leq -\infty) = P(\varnothing) = 0$$

Property 5 indicates that $F_X(x)$ is *continuous on the right.* This is the consequence of the definition (*2.4*).

**Table 2.1**

| $x$ | $(X \leq x)$ | $F_X(x)$ |
|---|---|---|
| $-1$ | $\varnothing$ | 0 |
| 0 | $(TTT)$ | $\frac{1}{8}$ |
| 1 | $\{TTT, TTH, THT, HTT\}$ | $\frac{4}{8} = \frac{1}{2}$ |
| 2 | $\{TTT, TTH, THT, HTT, HHT, HTH, THH\}$ | $\frac{7}{8}$ |
| 3 | $S$ | 1 |
| 4 | $S$ | 1 |

**EXAMPLE 2.3** Consider the r.v. $X$ defined in Example 2.2. Find and sketch the cdf $F_X(x)$ of $X$.

Table 2.1 gives $F_X(x) = P(X \leq x)$ for $x = -1, 0, 1, 2, 3, 4$. Since the value of $X$ must be an integer, the value of $F_X(x)$ for noninteger values of $x$ must be the same as the value of $F_X(x)$ for the nearest smaller integer value of $x$. The $F_X(x)$ is sketched in Fig. 2-3. Note that $F_X(x)$ has jumps at $x = 0, 1, 2, 3$, and that at each jump the upper value is the correct value for $F_X(x)$.

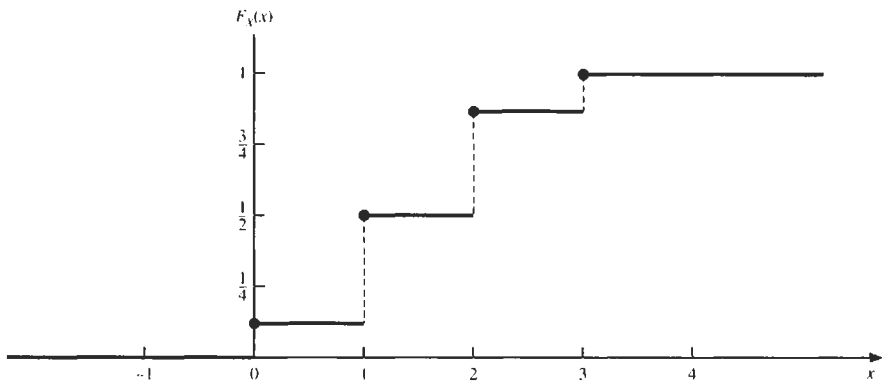



**Fig. 2-3**

### C. Determination of Probabilities from the Distribution Function:

From definition (*2.4*), we can compute other probabilities, such as $P(a < X \leq b)$, $P(X > a)$, and $P(X < b)$ (Prob. 2.6):

$$P(a < X \leq b) = F_X(b) - F_X(a) \tag{2.10}$$

$$P(X > a) = 1 - F_X(a) \tag{2.11}$$

$$P(X < b) = F_X(b^-) \qquad b^- = \lim_{0 < \varepsilon \to 0} b - \varepsilon \tag{2.12}$$

## 2.4 DISCRETE RANDOM VARIABLES AND PROBABILITY MASS FUNCTIONS

### A. Definition:

Let $X$ be a r.v. with cdf $F_X(x)$. If $F_X(x)$ changes values only in jumps (at most a countable number of them) and is constant between jumps—that is, $F_X(x)$ is a staircase function (see Fig. 2-3)— then $X$ is called a *discrete* random variable. Alternatively, $X$ is a discrete r.v. only if its range contains a finite or countably infinite number of points. The r.v. $X$ in Example 2.3 is an example of a discrete r.v.

### B. Probability Mass Functions:

Suppose that the jumps in $F_X(x)$ of a discrete r.v. $X$ occur at the points $x_1, x_2, \ldots$, where the sequence may be either finite or countably infinite, and we assume $x_i < x_j$ if $i < j$.

Then
$$F_X(x_i) - F_X(x_{i-1}) = P(X \leq x_i) - P(X \leq x_{i-1}) = P(X = x_i) \tag{2.13}$$

Let
$$p_X(x) = P(X = x) \tag{2.14}$$

The function $p_X(x)$ is called the *probability mass function* (pmf) of the discrete r.v. $X$.

***Properties of $p_X(x)$:***

1. $0 \leq p_X(x_k) \leq 1 \qquad k = 1, 2, \ldots$ (2.15)
2. $p_X(x) = 0 \qquad \text{if } x \neq x_k \ (k = 1, 2, \ldots)$ (2.16)
3. $\sum_k p_X(x_k) = 1$ (2.17)

The cdf $F_X(x)$ of a discrete r.v. $X$ can be obtained by

$$F_X(x) = P(X \leq x) = \sum_{x_k \leq x} p_X(x_k) \tag{2.18}$$

## 2.5 CONTINUOUS RANDOM VARIABLES AND PROBABILITY DENSITY FUNCTIONS

### A. Definition:

Let $X$ be a r.v. with cdf $F_X(x)$. If $F_X(x)$ is continuous and also has a derivative $dF_X(x)/dx$ which exists everywhere except at possibly a finite number of points and is piecewise continuous, then $X$ is called a *continuous* random variable. Alternatively, $X$ is a continuous r.v. only if its range contains an interval (either finite or infinite) of real numbers. Thus, if $X$ is a continuous r.v., then (Prob. 2.18)

$$P(X = x) = 0 \tag{2.19}$$

Note that this is an example of an event with probability 0 that is not necessarily the impossible event $\varnothing$.

In most applications, the r.v. is either discrete or continuous. But if the cdf $F_X(x)$ of a r.v. $X$ possesses features of both discrete and continuous r.v.'s, then the r.v. $X$ is called the *mixed* r.v. (Prob. 2.10).

### B. Probability Density Functions:

Let
$$f_X(x) = \frac{dF_X(x)}{dx} \tag{2.20}$$

The function $f_X(x)$ is called the *probability density function* (pdf) of the continuous r.v. $X$.

***Properties of $f_X(x)$:***

1. $f_X(x) \geq 0$ (2.21)
2. $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ (2.22)
3. $f_X(x)$ is piecewise continuous.
4. $P(a < X \leq b) = \int_a^b f_X(x)\,dx$ (2.23)

The cdf $F_X(x)$ of a continuous r.v. $X$ can be obtained by

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi \tag{2.24}$$

By Eq. (2.19), if $X$ is a continuous r.v., then

$$\begin{aligned} P(a < X \leq b) &= P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < b) \\ &= \int_a^b f_X(x)\,dx = F_X(b) - F_X(a) \end{aligned} \tag{2.25}$$

## 2.6 MEAN AND VARIANCE

### A. Mean:

The *mean* (or *expected value*) of a r.v. $X$, denoted by $\mu_X$ or $E(X)$, is defined by

$$\mu_X = E(X) = \begin{cases} \sum_k x_k\, p_X(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} x f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.26}$$

### B. Moment:

The $n$th *moment* of a r.v. $X$ is defined by

$$E(X^n) = \begin{cases} \sum_k x_k{}^n p_X(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} x^n f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.27}$$

Note that the mean of $X$ is the first moment of $X$.

### C. Variance:

The *variance* of a r.v. $X$, denoted by $\sigma_X{}^2$ or $\mathrm{Var}(X)$, is defined by

$$\sigma_X{}^2 = \mathrm{Var}(X) = E\{[X - E(X)]^2\} \tag{2.28}$$

Thus,

$$\sigma_X{}^2 = \begin{cases} \sum_k (x_k - \mu_X)^2 p_X(x_k) & X\text{: discrete} \\ \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx & X\text{: continuous} \end{cases} \tag{2.29}$$

Note from definition (*2.28*) that

$$\mathrm{Var}(X) \geq 0 \tag{2.30}$$

The *standard deviation* of a r.v. $X$, denoted by $\sigma_X$, is the positive square root of $\mathrm{Var}(X)$. Expanding the right-hand side of Eq. (*2.28*), we can obtain the following relation:

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 \tag{2.31}$$

which is a useful formula for determining the variance.

## 2.7 SOME SPECIAL DISTRIBUTIONS

In this section we present some important special distributions.

### A. Bernoulli Distribution:

A r.v. $X$ is called a *Bernoulli* r.v. with parameter $p$ if its pmf is given by

$$p_X(k) = P(X = k) = p^k(1 - p)^{1-k} \qquad k = 0, 1 \tag{2.32}$$

where $0 \leq p \leq 1$. By Eq. (*2.18*), the cdf $F_X(x)$ of the Bernoulli r.v. $X$ is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - p & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{2.33}$$

Figure 2-4 illustrates a Bernoulli distribution.

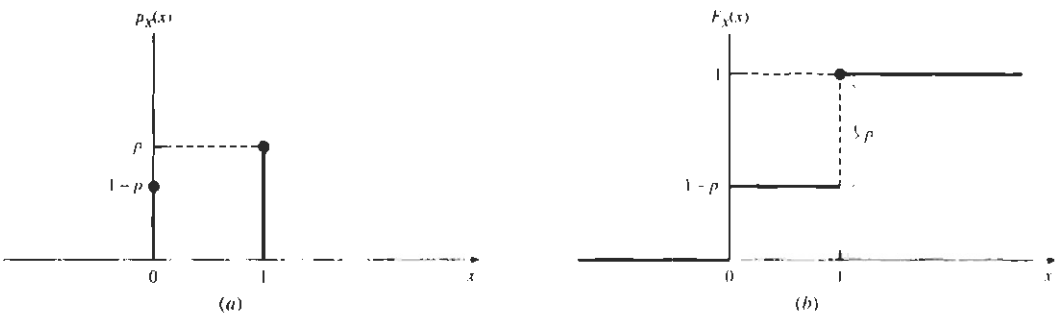



**Fig. 2-4** Bernoulli distribution.

The mean and variance of the Bernoulli r.v. $X$ are

$$\mu_X = E(X) = p \tag{2.34}$$

$$\sigma_X^2 = \mathrm{Var}(X) = p(1 - p) \tag{2.35}$$

A Bernoulli r.v. $X$ is associated with some experiment where an outcome can be classified as either a "success" or a "failure," and the probability of a success is $p$ and the probability of a failure is $1 - p$. Such experiments are often called *Bernoulli trials* (Prob. 1.61).

**B. Binomial Distribution:**

A r.v. $X$ is called a *binomial* r.v. with parameters $(n, p)$ if its pmf is given by

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, \ldots, n \tag{2.36}$$

where $0 \le p \le 1$ and

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

which is known as the binomial coefficient. The corresponding cdf of $X$ is

$$F_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} \qquad n \le x < n+1 \tag{2.37}$$

Figure 2-5 illustrates the binomial distribution for $n = 6$ and $p = 0.6$.

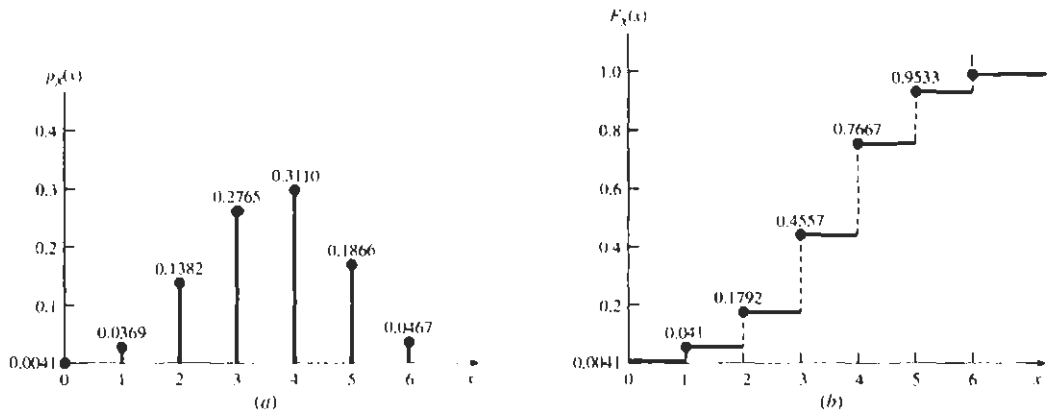



**Fig. 2-5** Binomial distribution with $n = 6$, $p = 0.6$.

The mean and variance of the binomial r.v. $X$ are (Prob. 2.28)

$$\mu_X = E(X) = np \tag{2.38}$$

$$\sigma_X^2 = \mathrm{Var}(X) = np(1-p) \tag{2.39}$$

A binomial r.v. $X$ is associated with some experiments in which $n$ independent Bernoulli trials are performed and $X$ represents the number of successes that occur in the $n$ trials. Note that a Bernoulli r.v. is just a binomial r.v. with paramcters $(1, p)$.

**C. Poisson Distribution:**

A r.v. $X$ is called a *Poisson* r.v. with parameter $\lambda\,(>0)$ if its pmf is given by

$$p_X(k) = P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots \tag{2.40}$$

The corresponding cdf of $X$ is

$$F_X(x) = e^{-\lambda} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \qquad n \le x < n+1 \tag{2.41}$$

Figure 2-6 illustrates the Poisson distribution for $\lambda = 3$.

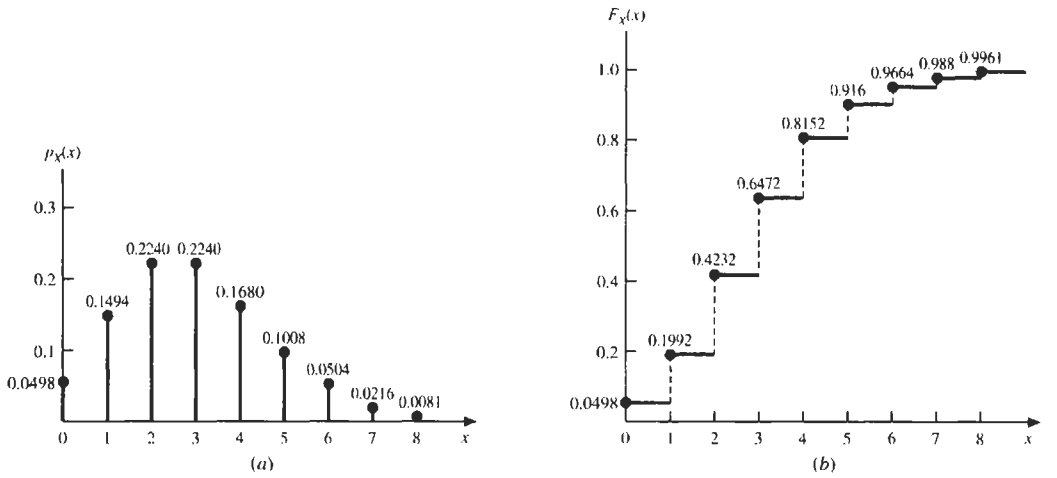



**Fig. 2-6** Poisson distribution with $\lambda = 3$.

The mean and variance of the Poisson r.v. $X$ are (Prob. 2.29)

$$\mu_X = E(X) = \lambda \tag{2.42}$$

$$\sigma_X^2 = \text{Var}(X) = \lambda \tag{2.43}$$

The Poisson r.v. has a tremendous range of applications in diverse areas because it may be used as an approximation for a binomial r.v. with parameters $(n, p)$ when $n$ is large and $p$ is small enough so that $np$ is of a moderate size (Prob. 2.40).

Some examples of Poisson r.v.'s include

1. The number of telephone calls arriving at a switching center during various intervals of time
2. The number of misprints on a page of a book
3. The number of customers entering a bank during various intervals of time

### D. Uniform Distribution:

A r.v. $X$ is called a *uniform* r.v. over $(a, b)$ if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases} \tag{2.44}$$

The corresponding cdf of $X$ is

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases} \tag{2.45}$$

Figure 2-7 illustrates a uniform distribution.

The mean and variance of the uniform r.v. $X$ are (Prob. 2.31)

$$\mu_X = E(X) = \frac{a+b}{2} \tag{2.46}$$

$$\sigma_X^2 = \text{Var}(X) = \frac{(b-a)^2}{12} \tag{2.47}$$

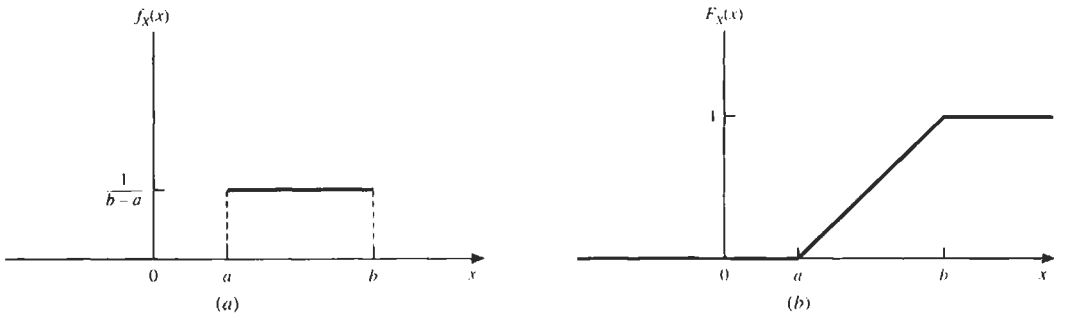



**Fig. 2-7** Uniform distribution over $(a, b)$.

A uniform r.v. $X$ is often used where we have no prior knowledge of the actual pdf and all continuous values in some range seem equally likely (Prob. 2.69).

### E. Exponential Distribution:

A r.v. X is called an *exponential* r.v. with parameter $\lambda$ $(>0)$ if its pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.48}$$

which is sketched in Fig. 2-8(*a*). The corresponding cdf of $X$ is

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{2.49}$$

which is sketched in Fig. 2-8(*b*).

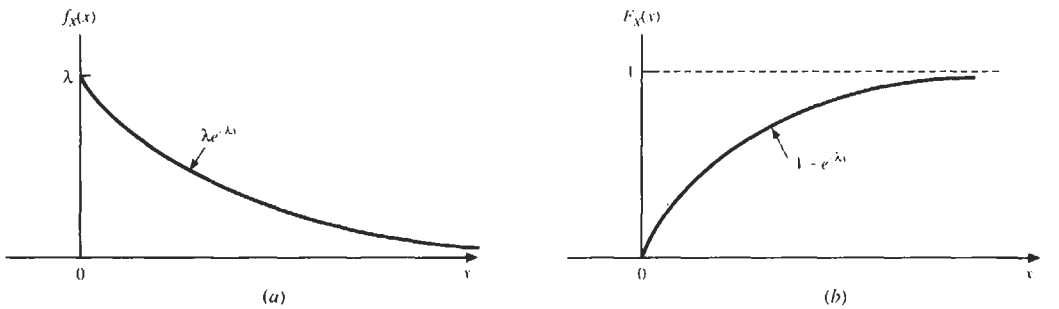



**Fig. 2-8** Exponential distribution.

The mean and variance of the exponential r.v. $X$ are (Prob. 2.32)

$$\mu_X = E(X) = \frac{1}{\lambda} \tag{2.50}$$

$$\sigma_X^2 = \mathrm{Var}(X) = \frac{1}{\lambda^2} \tag{2.51}$$

The most interesting property of the exponential distribution is its "memoryless" property. By this we mean that if the lifetime of an item is exponentially distributed, then an item which has been in use for some hours is as good as a new item with regard to the amount of time remaining until the item fails. The exponential distribution is the only distribution which possesses this property (Prob. 2.53).

### F. Normal (or Gaussian) Distribution:

A r.v. $X$ is called a *normal* (or *gaussian*) r.v. if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)} \tag{2.52}$$

The corresponding cdf of $X$ is

$$F_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{x} e^{-(\xi-\mu)^2/(2\sigma^2)}\, d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(x-\mu)/\sigma} e^{-\xi^2/2}\, d\xi \tag{2.53}$$

This integral cannot be evaluated in a closed form and must be evaluated numerically. It is convenient to use the function $\Phi(z)$, defined as

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi \tag{2.54}$$

to help us to evaluate the value of $F_X(x)$. Then Eq. (2.53) can be written as

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \tag{2.55}$$

Note that

$$\Phi(-z) = 1 - \Phi(z) \tag{2.56}$$

The function $\Phi(z)$ is tabulated in Table A (Appendix A). Figure 2-9 illustrates a normal distribution.

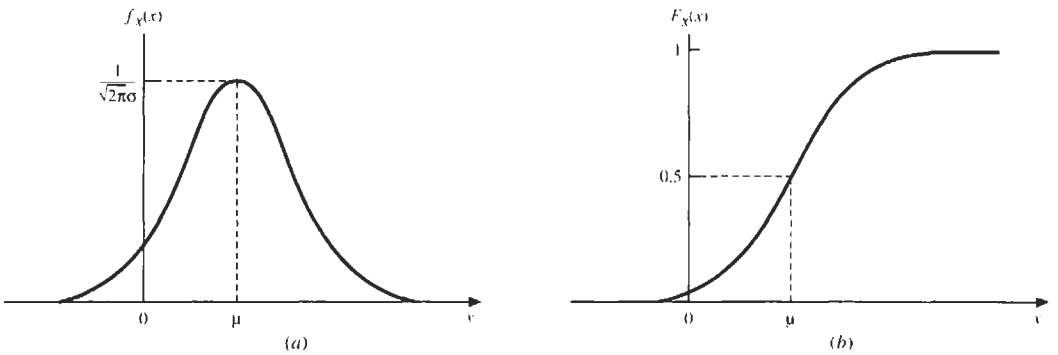



**Fig. 2-9** Normal distribution.

The mean and variance of the normal r.v. $X$ are (Prob. 2.33)

$$\mu_X = E(X) = \mu \tag{2.57}$$

$$\sigma_X^2 = \mathrm{Var}(X) = \sigma^2 \tag{2.58}$$

We shall use the notation $N(\mu; \sigma^2)$ to denote that $X$ is normal with mean $\mu$ and variance $\sigma^2$. A normal r.v. $Z$ with zero mean and unit variance—that is, $Z = N(0; 1)$—is called a *standard normal* r.v. Note that the cdf of the standard normal r.v. is given by Eq. (2.54). The normal r.v. is probably the most important type of continuous r.v. It has played a significant role in the study of random phenomena in nature. Many naturally occurring random phenomena are approximately normal. Another reason for the importance of the normal r.v. is a remarkable theorem called the *central limit theorem*. This theorem states that the sum of a large number of independent r.v.'s, under certain conditions, can be approximated by a normal r.v. (see Sec. 4.8C).

## 2.8 CONDITIONAL DISTRIBUTIONS

In Sec. 1.6 the conditional probability of an event $A$ given event $B$ is defined as

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad P(B) > 0$$

The conditional cdf $F_X(x \mid B)$ of a r.v. $X$ given event $B$ is defined by

$$F_X(x \mid B) = P(X \leq x \mid B) = \frac{P\{(X \leq x) \cap B\}}{P(B)} \tag{2.59}$$

The conditional cdf $F_X(x \mid B)$ has the same properties as $F_X(x)$. (See Prob. 1.37 and Sec. 2.3.) In particular,

$$F_X(-\infty \mid B) = 0 \qquad F_X(\infty \mid B) = 1 \tag{2.60}$$

$$P(a < X \leq b \mid B) = F_X(b \mid B) - F_X(a \mid B) \tag{2.61}$$

If $X$ is a discrete r.v., then the conditional pmf $p_X(x_k \mid B)$ is defined by

$$p_X(x_k \mid B) = P(X = x_k \mid B) = \frac{P\{(X = x_k) \cap B\}}{P(B)} \tag{2.62}$$

If $X$ is a continuous r.v., then the conditional pdf $f_X(x \mid B)$ is defined by

$$f_X(x \mid B) = \frac{dF_X(x \mid B)}{dx} \tag{2.63}$$

# Solved Problems

### RANDOM VARIABLES

**2.1.** Consider the experiment of throwing a fair die. Let $X$ be the r.v. which assigns 1 if the number that appears is even and 0 if the number that appears is odd.

(*a*) What is the range of $X$?

(*b*) Find $P(X = 1)$ and $P(X = 0)$.

The sample space $S$ on which $X$ is defined consists of 6 points which are equally likely:

$$S = \{1, 2, 3, 4, 5, 6\}$$

(*a*) The range of $X$ is $R_X = \{0, 1\}$.

(*b*) $(X = 1) = \{2, 4, 6\}$. Thus, $P(X = 1) = \frac{3}{6} = \frac{1}{2}$. Similarly, $(X = 0) = \{1, 3, 5\}$, and $P(X = 0) = \frac{1}{2}$.

**2.2.** Consider the experiment of tossing a coin three times (Prob. 1.1). Let $X$ be the r.v. giving the number of heads obtained. We assume that the tosses are independent and the probability of a head is $p$.

(*a*) What is the range of $X$?

(*b*) Find the probabilities $P(X = 0)$, $P(X = 1)$, $P(X = 2)$, and $P(X = 3)$.

The sample space $S$ on which $X$ is defined consists of eight sample points (Prob. 1.1):

$$S = \{HHH, HHT, \ldots, TTT\}$$

(*a*) The range of $X$ is $R_X = \{0, 1, 2, 3\}$.

(*b*) If $P(H) = p$, then $P(T) = 1 - p$. Since the tosses are independent, we have

$$P(X = 0) = P[\{TTT\}] = (1 - p)^3$$
$$P(X = 1) = P[\{HTT\}] + P[\{THT\}] + P[\{TTH\}] = 3(1 - p)^2 p$$
$$P(X = 2) = P[\{HHT\}] + P[\{HTH\}] + P[\{THH\}] = 3(1 - p)p^2$$
$$P(X = 3) = P[\{HHH\}] = p^3$$

**2.3.** An information source generates symbols at random from a four-letter alphabet $\{a, b, c, d\}$ with probabilities $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{4}$, and $P(c) = P(d) = \frac{1}{8}$. A coding scheme encodes these symbols into binary codes as follows:

| | |
|---|---|
| $a$ | 0 |
| $b$ | 10 |
| $c$ | 110 |
| $d$ | 111 |

Let $X$ be the r.v. denoting the length of the code, that is, the number of binary symbols (bits).

(*a*) What is the range of $X$?

(*b*) Assuming that the generations of symbols are independent, find the probabilities $P(X = 1)$, $P(X = 2)$, $P(X = 3)$, and $P(X > 3)$.

(*a*) The range of $X$ is $R_X = \{1, 2, 3\}$.

(*b*) $P(X = 1) = P[\{a\}] = P(a) = \frac{1}{2}$
$P(X = 2) = P[\{b\}] = P(b) = \frac{1}{4}$
$P(X = 3) = P[\{c, d\}] = P(c) + P(d) = \frac{1}{4}$
$P(X > 3) = P(\varnothing) = 0$

**2.4.** Consider the experiment of throwing a dart onto a circular plate with unit radius. Let $X$ be the r.v. representing the distance of the point where the dart lands from the origin of the plate. Assume that the dart always lands on the plate and that the dart is equally likely to land anywhere on the plate.

(*a*) What is the range of $X$?

(*b*) Find (i) $P(X < a)$ and (ii) $P(a < X < b)$, where $a < b \leq 1$.

(*a*) The range of $X$ is $R_X = \{x: 0 \leq x < 1\}$.

(*b*) (i) $(X < a)$ denotes that the point is inside the circle of radius $a$. Since the dart is equally likely to fall anywhere on the plate, we have (Fig. 2-10)

$$P(X < a) = \frac{\pi a^2}{\pi 1^2} = a^2$$

(ii) $(a < X < b)$ denotes the event that the point is inside the annular ring with inner radius $a$ and outer radius $b$. Thus, from Fig. 2-10, we have

$$P(a < X < b) = \frac{\pi(b^2 - a^2)}{\pi 1^2} = b^2 - a^2$$

## DISTRIBUTION FUNCTION

**2.5.** Verify Eq. (*2.6*).

Let $x_1 < x_2$. Then $(X \leq x_1)$ is a subset of $(X \leq x_2)$; that is, $(X \leq x_1) \subset (X \leq x_2)$. Then, by Eq. (*1.27*), we have

$$P(X \leq x_1) \leq P(X \leq x_2) \qquad \text{or} \qquad F_X(x_1) \leq F_X(x_2)$$

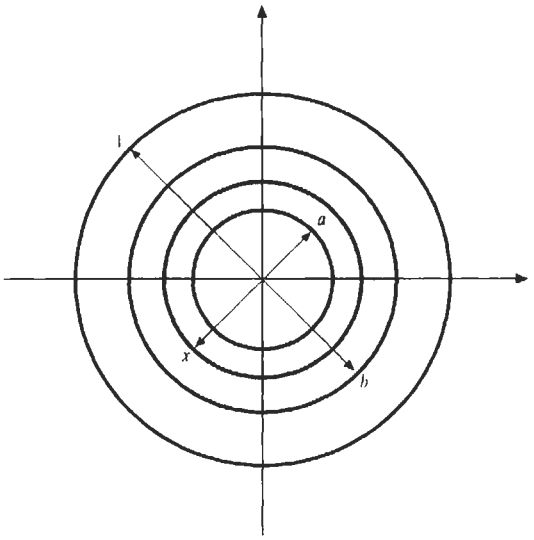



Fig. 2-10

**2.6.** Verify (*a*) Eq. (*2.10*); (*b*) Eq. (*2.11*); (*c*) Eq. (*2.12*).

(*a*) Since $(X \le b) = (X \le a) \cup (a < X \le b)$ and $(X \le a) \cap (a < X \le b) = \varnothing$, we have

$$P(X \le b) = P(X \le a) + P(a < X \le b)$$

or
$$F_X(b) = F_X(a) + P(a < X \le b)$$

Thus,
$$P(a < X \le b) = F_X(b) - F_X(a)$$

(*b*) Since $(X \le a) \cup (X > a) = S$ and $(X \le a) \cap (X > a) = \varnothing$, we have

$$P(X \le a) + P(X > a) = P(S) = 1$$

Thus,
$$P(X > a) = 1 - P(X \le a) = 1 - F_X(a)$$

(*c*) Now
$$P(X < b) = P[\lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} X \le b - \varepsilon] = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} P(X \le b - \varepsilon)$$
$$= \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} F_X(b - \varepsilon) = F_X(b^-)$$

**2.7.** Show that

(*a*) $P(a \le X \le b) = P(X = a) + F_X(b) - F_X(a)$ (*2.64*)

(*b*) $P(a < X < b) = F_X(b) - F_X(a) - P(X = b)$ (*2.65*)

(*c*) $P(a \le X < b) = P(X = a) + F_X(b) - F_X(a) - P(X = b)$ (*2.66*)

(*a*) Using Eqs. (*1.23*) and (*2.10*), we have

$$P(a \le X \le b) = P[(X = a) \cup (a < X \le b)]$$
$$= P(X = a) + P(a < X \le b)$$
$$= P(X = a) + F_X(b) - F_X(a)$$

(*b*) We have

$$P(a < X \le b) = P[(a < X < b) \cup (X = b)]$$
$$= P(a < X < b) + P(X = b)$$

Again using Eq. (2.10), we obtain

$$P(a < X < b) = P(a < X \le b) - P(X = b)$$
$$= F_X(b) - F_X(a) - P(X = b)$$

(c) Similarly,
$$P(a \le X \le b) = P[(a \le X < b) \cup (X = b)]$$
$$= P(a \le X < b) + P(X = b)$$

Using Eq. (2.64), we obtain

$$P(a \le X < b) = P(a \le X \le b) - P(X = b)$$
$$= P(X = a) + F_X(b) - F_X(a) - P(X = b)$$

**2.8.** Let $X$ be the r.v. defined in Prob. 2.3.

(a) Sketch the cdf $F_X(x)$ of $X$ and specify the type of $X$.

(b) Find (i) $P(X \le 1)$, (ii) $P(1 < X \le 2)$, (iii) $P(X > 1)$, and (iv) $P(1 \le X \le 2)$.

(a) From the result of Prob. 2.3 and Eq. (2.18), we have

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{3}{4} & 2 \le x < 3 \\ 1 & x \ge 3 \end{cases}$$

which is sketched in Fig. 2-11. The r.v. $X$ is a discrete r.v.

(b) (i) We see that

$$P(X \le 1) = F_X(1) = \tfrac{1}{2}$$

(ii) By Eq. (2.10),

$$P(1 < X \le 2) = F_X(2) - F_X(1) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

(iii) By Eq. (2.11),

$$P(X > 1) = 1 - F_X(1) = 1 - \tfrac{1}{2} = \tfrac{1}{2}$$

(iv) By Eq. (2.64),

$$P(1 \le X \le 2) = P(X = 1) + F_X(2) - F_X(1) = \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{3}{4}$$

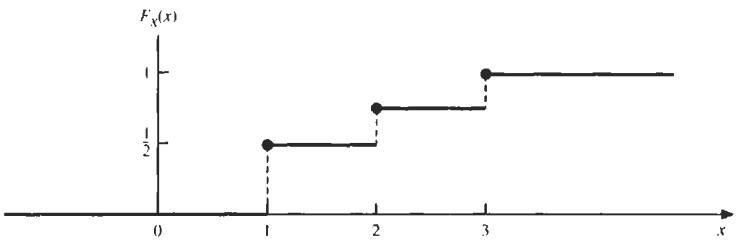



**Fig. 2-11**

**2.9.** Sketch the cdf $F_X(x)$ of the r.v. $X$ defined in Prob. 2.4 and specify the type of $X$.

From the result of Prob. 2.4, we have

$$F_X(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & 1 \le x \end{cases}$$

which is sketched in Fig. 2-12. The r.v. $X$ is a continuous r.v.

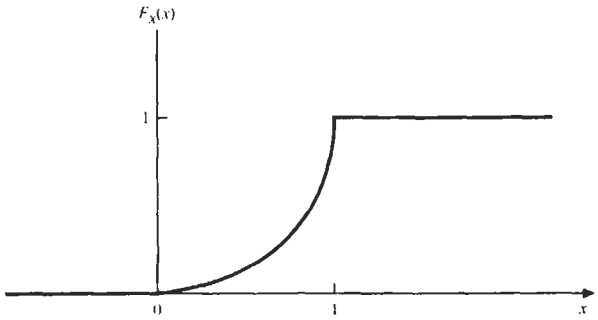



Fig. 2-12

**2.10.** Consider the function given by

$$F(x) = \begin{cases} 0 & x < 0 \\ x + \frac{1}{2} & 0 \le x < \frac{1}{2} \\ 1 & x \ge \frac{1}{2} \end{cases}$$

(*a*) Sketch $F(x)$ and show that $F(x)$ has the properties of a cdf discussed in Sec. 2.3B.

(*b*) If $X$ is the r.v. whose cdf is given by $F(x)$, find (i) $P(X \le \frac{1}{4})$, (ii) $P(0 < X \le \frac{1}{4})$, (iii) $P(X = 0)$, and (iv) $P(0 \le X \le \frac{1}{4})$.

(*c*) Specify the type of $X$.

(*a*) The function $F(x)$ is sketched in Fig. 2-13. From Fig. 2-13, we see that $0 \le F(x) \le 1$ and $F(x)$ is a nondecreasing function, $F(-\infty) = 0$, $F(\infty) = 1$, $F(0) = \frac{1}{2}$, and $F(x)$ is continuous on the right. Thus, $F(x)$ satisfies all the properties [Eqs. (2.5) to (2.9)] required of a cdf.

(*b*) (i) We have

$$P(X \le \tfrac{1}{4}) = F(\tfrac{1}{4}) = \tfrac{1}{4} + \tfrac{1}{2} = \tfrac{3}{4}$$

(ii) By Eq. (*2.10*),

$$P(0 < X \le \tfrac{1}{4}) = F(\tfrac{1}{4}) - F(0) = \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{1}{4}$$

(iii) By Eq. (*2.12*),

$$P(X = 0) = P(X \le 0) - P(X < 0) = F(0) - F(0^-) = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

(iv) By Eq. (*2.64*),

$$P(0 \le X \le \tfrac{1}{4}) = P(X = 0) + F(\tfrac{1}{4}) - F(0) = \tfrac{1}{2} + \tfrac{3}{4} - \tfrac{1}{2} = \tfrac{3}{4}$$

(*c*) The r.v. $X$ is a mixed r.v.

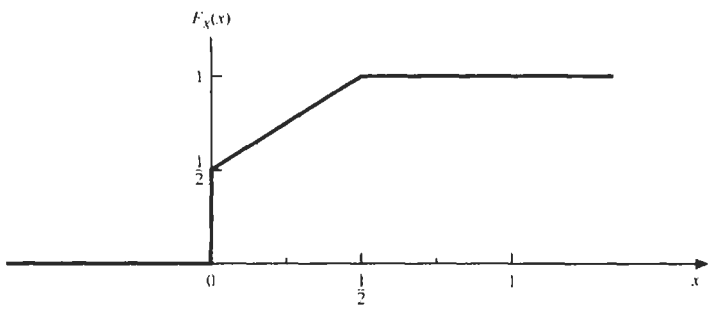



Fig. 2-13

**2.11.** Find the values of constants $a$ and $b$ such that

$$F(x) = \begin{cases} 1 - ae^{-x/b} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

is a valid cdf.

To satisfy property 1 of $F_X(x)$ $[0 \leq F_X(x) \leq 1]$, we must have $0 \leq a \leq 1$ and $b > 0$. Since $b > 0$, property 3 of $F_X(x)$ $[F_X(\infty) = 1]$ is satisfied. It is seen that property 4 of $F_X(x)$ $[F_X(-\infty) = 0]$ is also satisfied. For $0 \leq a \leq 1$ and $b > 0$, $F(x)$ is sketched in Fig. 2-14. From Fig. 2-14, we see that $F(x)$ is a nondecreasing function and continuous on the right, and properties 2 and 5 of $F_X(x)$ are satisfied. Hence, we conclude that $F(x)$ given is a valid cdf if $0 \leq a \leq 1$ and $b > 0$. Note that if $a = 0$, then the r.v. $X$ is a discrete r.v.; if $a = 1$, then $X$ is a continuous r.v.; and if $0 < a < 1$, then $X$ is a mixed r.v.

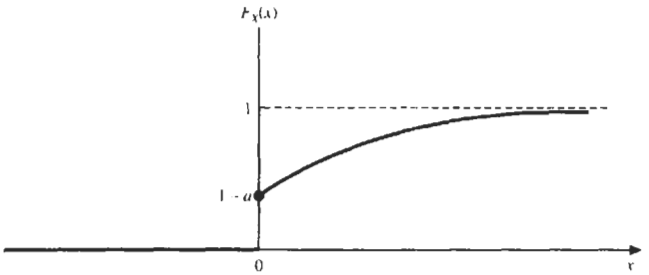



**Fig. 2-14**

## DISCRETE RANDOM VARIABLES AND PMF'S

**2.12.** Suppose a discrete r.v. $X$ has the following pmfs:

$$p_X(1) = \tfrac{1}{2} \qquad p_X(2) = \tfrac{1}{4} \qquad p_X(3) = \tfrac{1}{8} \qquad p_X(4) = \tfrac{1}{8}$$

(*a*) Find and sketch the cdf $F_X(x)$ of the r.v. $X$.

(*b*) Find (i) $P(X \leq 1)$, (ii) $P(1 < X \leq 3)$, (iii) $P(1 \leq X \leq 3)$.

(*a*) By Eq. (*2.18*), we obtain

$$F_X(x) = P(X \leq x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \frac{7}{8} & 3 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$$

which is sketched in Fig. 2-15.

(*b*) (i) By Eq. (*2.12*), we see that

$$P(X < 1) = F_X(1^-) = 0$$

(ii) By Eq. (*2.10*),

$$P(1 < X \leq 3) = F_X(3) - F_X(1) = \tfrac{7}{8} - \tfrac{1}{2} = \tfrac{3}{8}$$

(iii) By Eq. (*2.64*),

$$P(1 \leq X \leq 3) = P(X = 1) + F_X(3) - F_X(1) = \tfrac{1}{2} + \tfrac{7}{8} - \tfrac{1}{2} = \tfrac{7}{8}$$

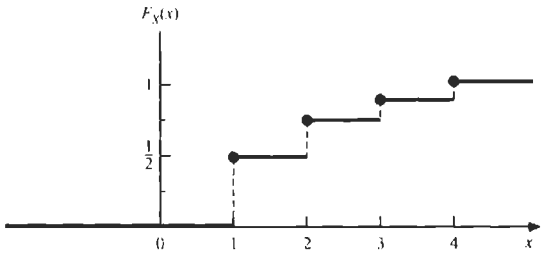



Fig. 2-15

**2.13.** (*a*) Verify that the function $p(x)$ defined by

$$p(x) = \begin{cases} \frac{3}{4}(\frac{1}{4})^x & x = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

is a pmf of a discrete r.v. $X$.

(*b*) Find (i) $P(X = 2)$, (ii) $P(X \leq 2)$, (iii) $P(X \geq 1)$.

(*a*) It is clear that $0 \leq p(x) \leq 1$ and

$$\sum_{i=0}^{\infty} p(i) = \frac{3}{4} \sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i = \frac{3}{4} \frac{1}{1 - \frac{1}{4}} = 1$$

Thus, $p(x)$ satisfies all properties of the pmf [Eqs. (*2.15*) to (*2.17*)] of a discrete r.v. $X$.

(*b*) (i) By definition (*2.14*),

$$P(X = 2) = p(2) = \tfrac{3}{4}(\tfrac{1}{4})^2 = \tfrac{3}{64}$$

(ii) By Eq. (*2.18*),

$$P(X \leq 2) = \sum_{i=0}^{2} p(i) = \tfrac{3}{4}(1 + \tfrac{1}{4} + \tfrac{1}{16}) = \tfrac{63}{64}$$

(iii) By Eq. (*1.25*),

$$P(X \geq 1) = 1 - P(X = 0) = 1 - p(0) = 1 - \tfrac{3}{4} = \tfrac{1}{4}$$

**2.14.** Consider the experiment of tossing an honest coin repeatedly (Prob. 1.35). Let the r.v. $X$ denote the number of tosses required until the first head appears.

(*a*) Find and sketch the pmf $p_X(x)$ and the cdf $F_X(x)$ of $X$.

(*b*) Find (i) $P(1 < X \leq 4)$, (ii) $P(X > 4)$.

(*a*) From the result of Prob. 1.35, the pmf of $X$ is given by

$$p_X(x) = p_X(k) = P(X = k) = (\tfrac{1}{2})^k \qquad k = 1, 2, \ldots$$

Then by Eq. (*2.18*),

$$F_X(x) = P(X \leq x) = \sum_{k=1}^{m \leq x} p_X(k) = \sum_{k=1}^{m \leq x} (\tfrac{1}{2})^k$$

or

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} & 1 \leq x < 2 \\ \frac{3}{4} & 2 \leq x < 3 \\ \vdots & \vdots \\ 1 - (\frac{1}{2})^n & n \leq x < n + 1 \\ \vdots & \vdots \end{cases}$$

These functions are sketched in Fig. 2-16.

(*b*) (i) By Eq. (*2.10*),

$$P(1 < X \leq 4) = F_X(4) - F_X(1) = \tfrac{15}{16} - \tfrac{1}{2} = \tfrac{7}{16}$$

(ii) By Eq. (*1.25*),

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \tfrac{15}{16} = \tfrac{1}{16}$$

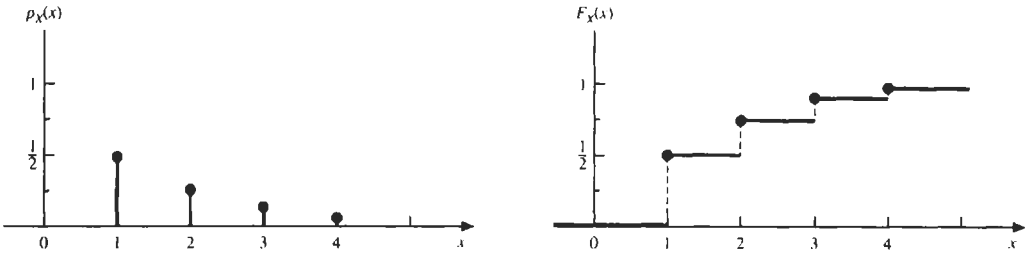



**Fig. 2-16**

**2.15.** Consider a sequence of Bernoulli trials with probability $p$ of success. This sequence is observed until the first success occurs. Let the r.v. $X$ denote the trial number on which this first success occurs. Then the pmf of $X$ is given by

$$p_X(x) = P(X = x) = (1 - p)^{x-1}p \qquad x = 1, 2, \ldots \tag{2.67}$$

because there must be $x - 1$ failures before the first success occurs on trial $x$. The r.v. $X$ defined by Eq. (*2.67*) is called a *geometric* r.v. with parameter $p$.

(*a*) Show that $p_X(x)$ given by Eq. (*2.67*) satisfies Eq. (*2.17*).

(*b*) Find the cdf $F_X(x)$ of $X$.

(*a*) Recall that for a geometric series, the sum is given by

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1 \tag{2.68}$$

Thus,

$$\sum_x p_X(x) = \sum_{i=1}^{\infty} p_X(i) = \sum_{i=1}^{\infty} (1 - p)^{i-1}p = \frac{p}{1 - (1 - p)} = \frac{p}{p} = 1$$

(*b*) Using Eq. (*2.68*), we obtain

$$P(X > k) = \sum_{i=k+1}^{\infty} (1 - p)^{i-1}p = \frac{(1 - p)^k p}{1 - (1 - p)} = (1 - p)^k \tag{2.69}$$

Thus,
$$P(X \leq k) = 1 - P(X > k) = 1 - (1 - p)^k \tag{2.70}$$

and
$$F_X(x) = P(X \leq x) = 1 - (1 - p)^x \qquad x = 1, 2, \ldots \tag{2.71}$$

Note that the r.v. $X$ of Prob. 2.14 is the geometric r.v. with $p = \frac{1}{2}$.

**2.16.** Let $X$ be a binomial r.v. with parameters $(n, p)$.

(*a*) Show that $p_X(x)$ given by Eq. (2.36) satisfies Eq. (*2.17*).

(*b*) Find $P(X > 1)$ if $n = 6$ and $p = 0.1$.

(*a*) Recall that the binomial expansion formula is given by

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \tag{2.72}$$

Thus, by Eq. (*2.36*),

$$\sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1 - p)^{n-k} = (p + 1 - p)^n = 1^n = 1$$

(*b*) Now

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{6}{0}(0.1)^0(0.9)^6 - \binom{6}{1}(0.1)^1(0.9)^5 \\ &= 1 - (0.9)^6 - 6(0.1)(0.9)^5 \approx 0.114 \end{aligned}$$

**2.17.** Let $X$ be a Poisson r.v. with parameter $\lambda$.

(*a*) Show that $p_X(x)$ given by Eq. (*2.40*) satisfies Eq. (*2.17*).

(*b*) Find $P(X > 2)$ with $\lambda = 4$.

(*a*) By Eq. (*2.40*),

$$\sum_{k=0}^{\infty} p_X(k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} = 1$$

(*b*) With $\lambda = 4$, we have

$$p_X(k) = e^{-4} \frac{4^k}{k!}$$

and

$$P(X \leq 2) = \sum_{k=0}^{2} p_X(k) = e^{-4}(1 + 4 + 8) \approx 0.238$$

Thus,

$$P(X > 2) = 1 - P(X \leq 2) \approx 1 - 0.238 = 0.762$$

## CONTINUOUS RANDOM VARIABLES AND PDF'S

**2.18.** Verify Eq. (*2.19*).

From Eqs. (*1.27*) and (*2.10*), we have

$$P(X = x) \leq P(x - \varepsilon < X \leq x) = F_X(x) - F_X(x - \varepsilon)$$

for any $\varepsilon \geq 0$. As $F_X(x)$ is continuous, the right-hand side of the above expression approaches 0 as $\varepsilon \to 0$. Thus, $P(X = x) = 0$.

**2.19.** The pdf of a continuous r.v. $X$ is given by

$$f_X(x) = \begin{cases} \frac{1}{3} & 0 < x < 1 \\ \frac{2}{3} & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the corresponding cdf $F_X(x)$ and sketch $f_X(x)$ and $F_X(x)$.

By Eq. (2.24), the cdf of $X$ is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\int_0^x \tfrac{1}{3}\, d\xi = \frac{x}{3} & 0 \le x < 1 \\ \displaystyle\int_0^1 \tfrac{1}{3}\, d\xi + \int_1^x \tfrac{2}{3}\, d\xi = \tfrac{2}{3}x - \tfrac{1}{3} & 1 \le x < 2 \\ \displaystyle\int_0^1 \tfrac{1}{3}\, d\xi + \int_1^2 \tfrac{2}{3}\, d\xi = 1 & 2 \le x \end{cases}$$

The functions $f_X(x)$ and $F_X(x)$ are sketched in Fig. 2-17.

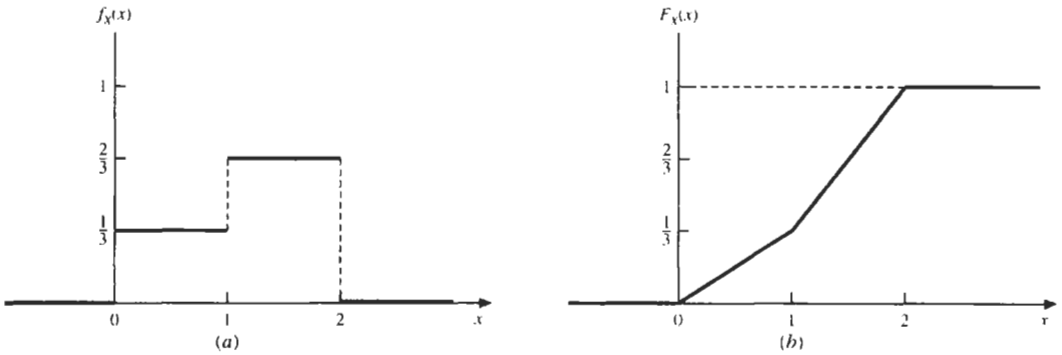



**Fig. 2-17**

**2.20.** Let $X$ be a continuous r.v. $X$ with pdf

$$f_X(x) = \begin{cases} kx & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Determine the value of $k$ and sketch $f_X(x)$.

(*b*) Find and sketch the corresponding cdf $F_X(x)$.

(*c*) Find $P(\frac{1}{4} < X \le 2)$.

(*a*) By Eq. (2.21), we must have $k > 0$, and by Eq. (2.22),

$$\int_0^1 kx\, dx = \frac{k}{2} = 1$$

Thus, $k = 2$ and

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

which is sketched in Fig. 2-18(*a*).

(*b*) By Eq. (2.24), the cdf of $X$ is given by

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \displaystyle\int_0^x 2\xi\, d\xi = x^2 & 0 \le x < 1 \\ \displaystyle\int_0^1 2\xi\, d\xi = 1 & 1 \le x \end{cases}$$

which is sketched in Fig. 2-18(*b*).

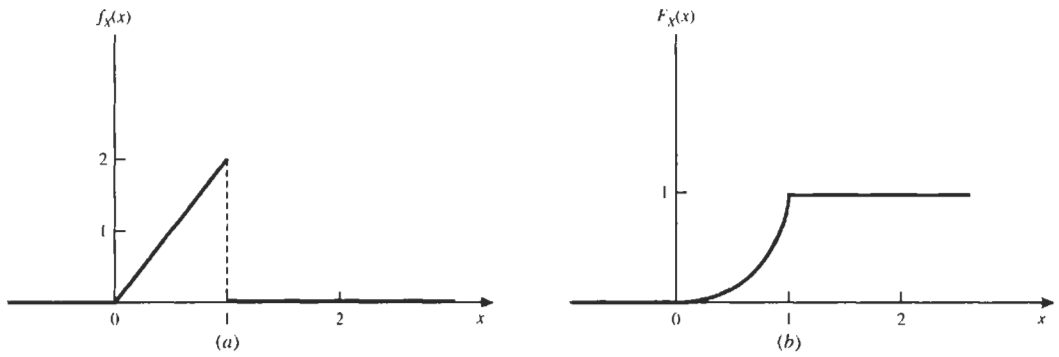



**Fig. 2-18**

(*c*) By Eq. (2.25),

$$P(\tfrac{1}{4} < X \leq 2) = F_X(2) - F_X(\tfrac{1}{4}) = 1 - (\tfrac{1}{4})^2 = \tfrac{15}{16}$$

**2.21.** Show that the pdf of a normal r.v. $X$ given by Eq. (2.52) satisfies Eq. (2.22).

From Eq. (2.52),

$$\int_{-\infty}^{\infty} f_x(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx$$

Let $y = (x-\mu)/(\sqrt{2}\,\sigma)$. Then $dx = \sqrt{2}\,\sigma\,dy$ and

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy$$

Let

$$\int_{-\infty}^{\infty} e^{-y^2}\,dy = I$$

Then

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\,dx\right]\left[\int_{-\infty}^{\infty} e^{-y^2}\,dy\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Letting $x = r\cos\theta$ and $y = r\sin\theta$ (that is, using polar coordinates), we have

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = 2\pi\int_0^{\infty} e^{-r^2} r\,dr = \pi$$

Thus,

$$I = \int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi} \tag{2.73}$$

and

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-y^2}\,dy = \frac{1}{\sqrt{\pi}}\sqrt{\pi} = 1$$

**2.22.** Consider a function

$$f(x) = \frac{1}{\sqrt{\pi}}\,e^{(-x^2+x-a)} \qquad -\infty < x < \infty$$

Find the value of $a$ such that $f(x)$ is a pdf of a continuous r.v. $X$.

$$f(x) = \frac{1}{\sqrt{\pi}} e^{(-x^2+x-a)} = \frac{1}{\sqrt{\pi}} e^{-(x^2-x+1/4+a-1/4)}$$
$$= \left[\frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\right] e^{-(a-1/4)}$$

If $f(x)$ is a pdf of a continuous r.v. $X$, then by Eq. (2.22), we must have

$$\int_{-\infty}^{\infty} f(x)\,dx = e^{-(a-1/4)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\,dx = 1$$

Now by Eq. (2.52), the pdf of $N(\frac{1}{2}; \frac{1}{2})$ is $\frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}$. Thus,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-(x-1/2)^2}\,dx = 1 \qquad \text{and} \qquad \int_{-\infty}^{\infty} f(x)\,dx = e^{-(a-1/4)} = 1$$

from which we obtain $a = \frac{1}{4}$.

**2.23.** A r.v. $X$ is called a *Rayleigh* r.v. if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.74}$$

(*a*) Determine the corresponding cdf $F_X(x)$.

(*b*) Sketch $f_X(x)$ and $F_X(x)$ for $\sigma = 1$.

(*a*) By Eq. (2.24), the cdf of $X$ is

$$F_X(x) = \int_0^x \frac{\xi}{\sigma^2} e^{-\xi^2/(2\sigma^2)}\,d\xi \qquad x \geq 0$$

Let $y = \xi^2/(2\sigma^2)$. Then $dy = (1/\sigma^2)\xi\,d\xi$, and

$$F_X(x) = \int_0^{x^2/(2\sigma^2)} e^{-y}\,dy = 1 - e^{-x^2/(2\sigma^2)} \tag{2.75}$$

(*b*) With $\sigma = 1$, we have

$$f_X(x) = \begin{cases} xe^{-x^2/2} & x > 0 \\ 0 & x < 0 \end{cases}$$

and

$$F_X(x) = \begin{cases} 1 - e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

These functions are sketched in Fig. 2-19.

**2.24.** A r.v. $X$ is called a *gamma* r.v. with parameter $(\alpha, \lambda)$ ($\alpha > 0$ and $\lambda > 0$) if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.76}$$

where $\Gamma(\alpha)$ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-x} x^{\alpha-1}\,dx \qquad \alpha > 0 \tag{2.77}$$

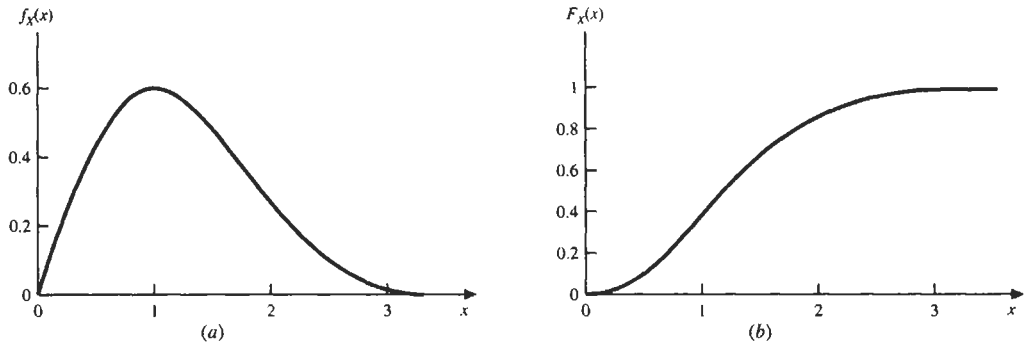



**Fig. 2-19** Rayleigh distribution with $\sigma = 1$.

(*a*) Show that the gamma function has the following properties:

1. $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha) \qquad \alpha > 0$ (*2.78*)
2. $\Gamma(k + 1) = k! \qquad k\ (\geq 0)$: integer (*2.79*)
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ (*2.80*)

(*b*) Show that the pdf given by Eq. (*2.76*) satisfies Eq. (*2.22*).

(*c*) Plot $f_X(x)$ for $(\alpha, \lambda) = (1, 1), (2, 1)$, and $(5, 2)$.

(*a*) Integrating Eq. (*2.77*) by parts ($u = x^{\alpha-1}$, $dv = e^{-x}\,dx$), we obtain

$$\Gamma(\alpha) = -e^{-x}x^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha - 1)x^{\alpha-2}\,dx$$

$$= (\alpha - 1)\int_0^\infty e^{-x}x^{\alpha-2}\,dx = (\alpha - 1)\Gamma(\alpha - 1) \tag{2.81}$$

Replacing $\alpha$ by $\alpha + 1$ in Eq. (*2.81*), we get Eq. (*2.78*).

Next, by applying Eq. (*2.78*) repeatedly using an integral value of $\alpha$, say $\alpha = k$, we obtain

$$\Gamma(k + 1) = k\Gamma(k) = k(k - 1)\Gamma(k - 1) = k(k - 1)\cdots(2)\Gamma(1)$$

Since

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1$$

it follows that $\Gamma(k + 1) = k!$. Finally, by Eq. (*2.77*),

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty e^{-x}x^{-1/2}\,dx$$

Let $y = x^{1/2}$. Then $dy = \frac{1}{2}x^{-1/2}\,dx$, and

$$\Gamma(\tfrac{1}{2}) = 2\int_0^\infty e^{-y^2}\,dy = \int_{-\infty}^\infty e^{-y^2}\,dy = \sqrt{\pi}$$

in view of Eq. (*2.73*).

(*b*) Now

$$\int_{-\infty}^\infty f_X(x)\,dx = \int_0^\infty \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-\lambda x}x^{\alpha-1}\,dx$$

Let $y = \lambda x$. Then $dy = \lambda\,dx$ and

$$\int_{-\infty}^\infty f_X(x)\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha}\int_0^\infty e^{-y}y^{\alpha-1}\,dy = \frac{\lambda^\alpha}{\Gamma(\alpha)\lambda^\alpha}\Gamma(\alpha) = 1$$

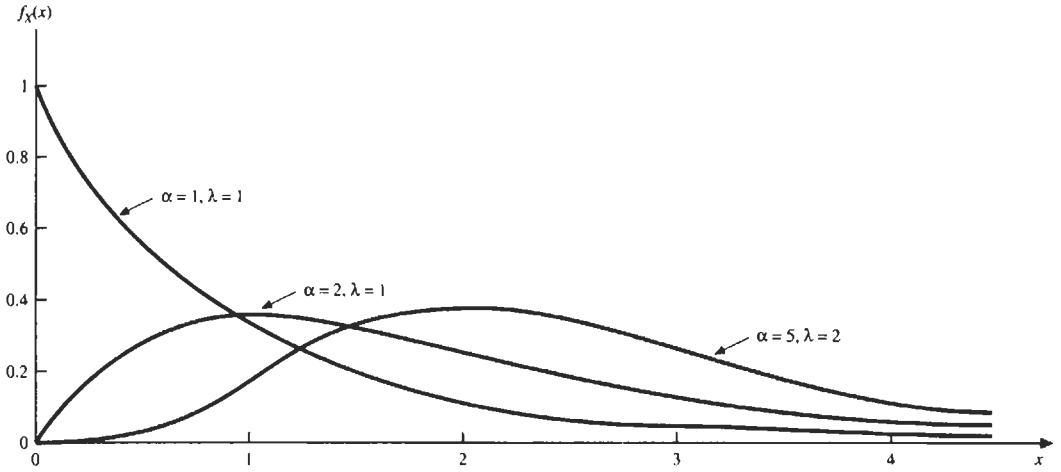



**Fig. 2-20** Gamma distributions for selected values of $\alpha$ and $\lambda$.

(*c*) The pdf's $f_X(x)$ with $(\alpha, \lambda) = (1, 1), (2, 1)$, and $(5, 2)$ are plotted in Fig. 2-20.

Note that when $\alpha = 1$, the gamma r.v. becomes an exponential r.v. with parameter $\lambda$ [Eq. (*2.48*)].

## MEAN AND VARIANCE

**2.25.** Consider a discrete r.v. $X$ whose pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Plot $p_X(x)$ and find the mean and variance of $X$.

(*b*) Repeat (*a*) if the pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -2, 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) The pmf $p_X(x)$ is plotted in Fig. 2-21(*a*). By Eq. (*2.26*), the mean of $X$ is

$$\mu_X = E(X) = \tfrac{1}{3}(-1 + 0 + 1) = 0$$

By Eq. (*2.29*), the variance of $X$ is

$$\sigma_X^2 = \mathrm{Var}(X) = E[(X - \mu_X)^2] = E(X^2) = \tfrac{1}{3}[(-1)^2 + (0)^2 + (1)^2] = \tfrac{2}{3}$$

(*b*) The pmf $p_X(x)$ is plotted in Fig. 2-21(*b*). Again by Eqs. (*2.26*) and (*2.29*), we obtain

$$\mu_X = E(X) = \tfrac{1}{3}(-2 + 0 + 2) = 0$$

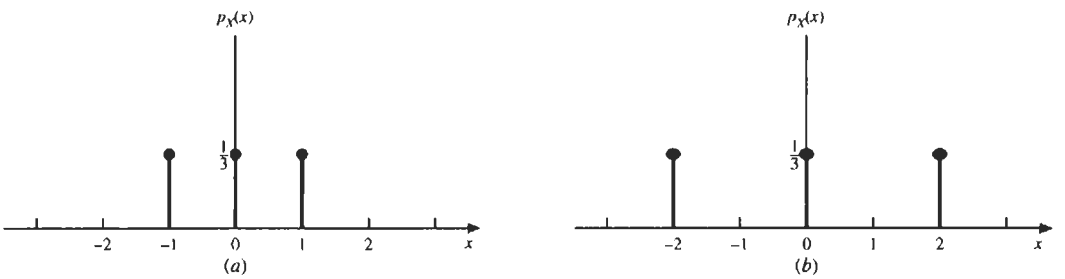



**Fig. 2-21**

$$\sigma_X^2 = \text{Var}(X) = \tfrac{1}{3}[(-2)^2 + (0)^2 + (2)^2] = \tfrac{8}{3}$$

Note that the variance of $X$ is a measure of the spread of a distribution about its mean.

**2.26.** Let a r.v. $X$ denote the outcome of throwing a fair die. Find the mean and variance of $X$.

Since the die is fair, the pmf of $X$ is

$$p_X(x) = p_X(k) = \tfrac{1}{6} \qquad k = 1, 2, \ldots, 6$$

By Eqs. (*2.26*) and (*2.29*), the mean and variance of $X$ are

$$\mu_X = E(X) = \tfrac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = \tfrac{7}{2} = 3.5$$
$$\sigma_X^2 = \tfrac{1}{6}[(1 - \tfrac{7}{2})^2 + (2 - \tfrac{7}{2})^2 + (3 - \tfrac{7}{2})^2 + (4 - \tfrac{7}{2})^2 + (5 - \tfrac{7}{2})^2 + (6 - \tfrac{7}{2})^2] = \tfrac{35}{12}$$

Alternatively, the variance of $X$ can be found as follows:

$$E(X^2) = \tfrac{1}{6}(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2) = \tfrac{91}{6}$$

Hence, by Eq. (*2.31*),

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \tfrac{91}{6} - (\tfrac{7}{2})^2 = \tfrac{35}{12}$$

**2.27.** Find the mean and variance of the geometric r.v. $X$ defined by Eq. (*2.67*) (Prob. 2.15).

To find the mean and variance of a geometric r.v. $X$, we need the following results about the sum of a geometric series and its first and second derivatives. Let

$$g(r) = \sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r} \qquad |r| < 1 \tag{2.82}$$

Then

$$g'(r) = \frac{dg(r)}{dr} = \sum_{n=1}^{\infty} anr^{n-1} = \frac{a}{(1 - r)^2} \tag{2.83}$$

$$g''(r) = \frac{d^2g(r)}{dr^2} = \sum_{n=2}^{\infty} an(n - 1)r^{n-2} = \frac{2a}{(1 - r)^3} \tag{2.84}$$

By Eqs. (*2.26*) and (*2.67*), and letting $q = 1 - p$, the mean of $X$ is given by

$$\mu_X = E(X) = \sum_{x=1}^{\infty} xq^{x-1}p = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} \tag{2.85}$$

where Eq. (*2.83*) is used with $a = p$ and $r = q$.

To find the variance of $X$, we first find $E[X(X - 1)]$. Now,

$$E[X(X - 1)] = \sum_{x=1}^{\infty} x(x - 1)q^{x-1}p = \sum_{x=2}^{\infty} pqx(x - 1)q^{x-2}$$
$$= \frac{2pq}{(1 - q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2} = \frac{2(1 - p)}{p^2} \tag{2.86}$$

where Eq. (*2.84*) is used with $a = pq$ and $r = q$.

Since $E[X(X - 1)] = E(X^2 - X) = E(X^2) - E(X)$, we have

$$E(X^2) = E[X(X - 1)] + E(X) = \frac{2(1 - p)}{p^2} + \frac{1}{p} = \frac{2 - p}{p^2} \tag{2.87}$$

Then by Eq. (*2.31*), the variance of $X$ is

$$\sigma_X^2 = \text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2 - p}{p^2} - \frac{1}{p^2} = \frac{1 - p}{p^2} \tag{2.88}$$

**2.28.** Let $X$ be a binomial r.v. with parameters $(n, p)$. Verify Eqs. (*2.38*) and (*2.39*).

By Eqs. (2.26) and (2.36), and letting $q = 1 - p$, we have

$$E(X) = \sum_{k=0}^{n} kp_X(k) = \sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k\frac{n!}{(n-k)!\,k!}p^k q^{n-k}$$

$$= np\sum_{k=1}^{n}\frac{(n-1)!}{(n-k)!\,(k-1)!}p^{k-1}q^{n-k}$$

Letting $i = k - 1$ and using Eq. (2.72), we obtain

$$E(X) = np\sum_{i=0}^{n-1}\frac{(n-1)!}{(n-1-i)!\,i!}p^i q^{n-1-i}$$

$$= np\sum_{i=0}^{n-1}\binom{n-1}{i}p^i q^{n-1-i}$$

$$= np(p+q)^{n-1} = np(1)^{n-1} = np$$

Next,

$$E[X(X-1)] = \sum_{k=0}^{n} k(k-1)p_X(k) = \sum_{k=0}^{n} k(k-1)\binom{n}{k}p^k q^{n-k}$$

$$= \sum_{k=0}^{n} k(k-1)\frac{n!}{(n-k)!\,k!}p^k q^{n-k}$$

$$= n(n-1)p^2\sum_{k=2}^{n}\frac{(n-2)!}{(n-k)!\,(k-2)!}p^{k-2}q^{n-k}$$

Similarly, letting $i = k - 2$ and using Eq. (2.72), we obtain

$$E[X(X-1)] = n(n-1)p^2\sum_{i=0}^{n-2}\frac{(n-2)!}{(n-2-i)!\,i!}p^i q^{n-2-i}$$

$$= n(n-1)p^2\sum_{i=0}^{n-2}\binom{n-2}{i}p^i q^{n-2-i}$$

$$= n(n-1)p^2(p+q)^{n-2} = n(n-1)p^2$$

Thus,

$$E(X^2) = E[X(X-1)] + E(X) = n(n-1)p^2 + np \tag{2.89}$$

and by Eq. (2.31),

$$\sigma_X^2 = \mathrm{Var}(x) = n(n-1)p^2 + np - (np)^2 = np(1-p)$$

**2.29.** Let $X$ be a Poisson r.v. with parameter $\lambda$. Verify Eqs. (2.42) and (2.43).

By Eqs. (2.26) and (2.40),

$$E(X) = \sum_{k=0}^{\infty} kp_X(k) = \sum_{k=0}^{\infty} ke^{-\lambda}\frac{\lambda^k}{k!} = 0 + \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

Next,

$$E[X(X-1)] = \sum_{k=0}^{\infty} k(k-1)e^{-\lambda}\frac{\lambda^k}{k!} = \lambda^2 e^{-\lambda}\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!}$$

$$= \lambda^2 e^{-\lambda}\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = \lambda^2 e^{-\lambda}e^{\lambda} = \lambda^2$$

Thus,

$$E(X^2) = E[X(X-1)] + E(X) = \lambda^2 + \lambda \tag{2.90}$$

and by Eq. (*2.31*),

$$\sigma_X^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

**2.30.** Find the mean and variance of the r.v. $X$ of Prob. 2.20.

From Prob. 2.20, the pdf of $X$ is

$$f_X(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

By Eq. (*2.26*), the mean of $X$ is

$$\mu_X = E(X) = \int_0^1 x(2x)\,dx = 2\,\frac{x^3}{3}\bigg|_0^1 = \tfrac{2}{3}$$

By Eq. (*2.27*), we have

$$E(X^2) = \int_0^1 x^2(2x)\,dx = 2\,\frac{x^4}{4}\bigg|_0^1 = \tfrac{1}{2}$$

Thus, by Eq. (*2.31*), the variance of $X$ is

$$\sigma_X^2 = \mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \tfrac{1}{2} - (\tfrac{2}{3})^2 = \tfrac{1}{18}$$

**2.31.** Let $X$ be a uniform r.v. over $(a, b)$. Verify Eqs. (*2.46*) and (*2.47*).

By Eqs. (*2.44*) and (*2.26*), the mean of $X$ is

$$\mu_X = E(X) = \int_a^b x\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\bigg|_a^b = \frac{1}{2}\,\frac{b^2-a^2}{b-a} = \tfrac{1}{2}(b+a)$$

By Eq. (*2.27*), we have

$$E(X^2) = \int_a^b x^2\,\frac{1}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^3}{3}\bigg|_a^b = \tfrac{1}{3}(b^2+ab+a^2) \tag{2.91}$$

Thus, by Eq. (*2.31*), the variance of $X$ is

$$\begin{aligned}\sigma_X^2 &= \mathrm{Var}(X) = E(X^2) - [E(X)]^2 \\ &= \tfrac{1}{3}(b^2+ab+a^2) - \tfrac{1}{4}(b+a)^2 = \tfrac{1}{12}(b-a)^2\end{aligned}$$

**2.32.** Let $X$ be an exponential r.v. $X$ with parameter $\lambda$. Verify Eqs. (*2.50*) and (*2.51*).

By Eqs. (*2.48*) and (*2.26*), the mean of $X$ is

$$\mu_X = E(X) = \int_0^\infty x\lambda e^{-\lambda x}\,dx$$

Integrating by parts ($u = x$, $dv = \lambda e^{-\lambda x}\,dx$) yields

$$E(X) = -xe^{-\lambda x}\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx = \frac{1}{\lambda}$$

Next, by Eq. (*2.27*),

$$E(X^2) = \int_0^\infty x^2\lambda e^{-\lambda x}\,dx$$

Again integrating by parts ($u = x^2$, $dv = \lambda e^{-\lambda x}\,dx$), we obtain

$$E(X^2) = -x^2e^{-\lambda x}\Big|_0^\infty + 2\int_0^\infty xe^{-\lambda x}\,dx = \frac{2}{\lambda^2} \tag{2.92}$$

Thus, by Eq. (*2.31*), the variance of $X$ is

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

**2.33.** Let $X = N(\mu; \sigma^2)$. Verify Eqs. (*2.57*) and (*2.58*).

Using Eqs. (*2.52*) and (*2.26*), we have

$$\mu_X = E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

Writing $x$ as $(x - \mu) + \mu$, we have

$$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu) e^{-(x-\mu)^2/(2\sigma^2)}\, dx + \mu \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

Letting $y = x - \mu$ in the first integral, we obtain

$$E(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y e^{-y^2/(2\sigma^2)}\, dy + \mu \int_{-\infty}^{\infty} f_X(x)\, dx$$

The first integral is zero, since its integrand is an odd function. Thus, by the property of pdf Eq. (*2.22*), we get

$$\mu_X = E(X) = \mu$$

Next, by Eq. (*2.29*),

$$\sigma_X^2 = E[(X-\mu)^2] = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

From Eqs. (*2.22*) and (*2.52*), we have

$$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sigma\sqrt{2\pi}$$

Differentiating with respect to $\sigma$, we obtain

$$\int_{-\infty}^{\infty} \frac{(x-\mu)^2}{\sigma^3} e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sqrt{2\pi}$$

Multiplying both sides by $\sigma^2/\sqrt{2\pi}$, we have

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/(2\sigma^2)}\, dx = \sigma^2$$

Thus,
$$\sigma_X^2 = \mathrm{Var}(X) = \sigma^2$$

**2.34.** Find the mean and variance of a Rayleigh r.v. defined by Eq. (*2.74*) (Prob. 2.23).

Using Eqs. (2.74) and (2.26), we have

$$\mu_X = E(X) = \int_0^{\infty} x \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}\, dx = \frac{1}{\sigma^2} \int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\, dx$$

Now the variance of $N(0; \sigma^2)$ is given by

$$\frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x^2 e^{-x^2/(2\sigma^2)}\, dx = \sigma^2$$

Since the integrand is an even function, we have

$$\frac{1}{\sqrt{2\pi}\,\sigma}\int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\,dx = \tfrac{1}{2}\sigma^2$$

or

$$\int_0^{\infty} x^2 e^{-x^2/(2\sigma^2)}\,dx = \tfrac{1}{2}\sqrt{2\pi}\,\sigma^3 = \sqrt{\frac{\pi}{2}}\,\sigma^3$$

Then

$$\mu_X = E(X) = \frac{1}{\sigma^2}\sqrt{\frac{\pi}{2}}\,\sigma^3 = \sqrt{\frac{\pi}{2}}\,\sigma \tag{2.93}$$

Next,

$$E(X^2) = \int_0^{\infty} x^2 \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)}\,dx = \frac{1}{\sigma^2}\int_0^{\infty} x^3 e^{-x^2/(2\sigma^2)}\,dx$$

Let $y = x^2/(2\sigma^2)$. Then $dy = x\,dx/\sigma^2$, and so

$$E(X^2) = 2\sigma^2 \int_0^{\infty} y e^{-y}\,dy = 2\sigma^2 \tag{2.94}$$

Hence, by Eq. (*2.31*),

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = \left(2 - \frac{\pi}{2}\right)\sigma^2 \approx 0.429\sigma^2 \tag{2.95}$$

**2.35.** Consider a continuous r.v. $X$ with pdf $f_X(x)$. If $f_X(x) = 0$ for $x < 0$, then show that, for any $a > 0$,

$$P(X \geq a) \leq \frac{\mu_X}{a} \tag{2.96}$$

where $\mu_X = E(X)$. This is known as the *Markov inequality*.

From Eq. (*2.23*),

$$P(X \geq a) = \int_a^{\infty} f_X(x)\,dx$$

Since $f_X(x) = 0$ for $x < 0$,

$$\mu_X = E(X) = \int_0^{\infty} x f_X(x)\,dx \geq \int_a^{\infty} x f_X(x)\,dx \geq a\int_a^{\infty} f_X(x)\,dx$$

Hence,

$$\int_a^{\infty} f_X(x)\,dx = P(X \geq a) \leq \frac{\mu_X}{a}$$

**2.36.** For any $a > 0$, show that

$$P(|X - \mu_X| \geq a) \leq \frac{\sigma_X^2}{a^2} \tag{2.97}$$

where $\mu_X$ and $\sigma_X^2$ are the mean and variance of $X$, respectively. This is known as the *Chebyshev inequality*.

From Eq. (*2.23*),

$$P(|X - \mu_X| \geq a) = \int_{-\infty}^{\mu_X - a} f_X(x)\,dx + \int_{\mu_X + a}^{\infty} f_X(x)\,dx = \int_{|x-\mu_X| \geq a} f_X(x)\,dx$$

By Eq. (*2.29*),

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\,dx \geq \int_{|x-\mu_X| \geq a} (x - \mu_X)^2 f_X(x)\,dx \geq a^2 \int_{|x-\mu_X| \geq a} f_X(x)\,dx$$

Hence, $$\int_{|x-\mu_X|\geq a} f_X(x)\,dx \leq \frac{\sigma_X^2}{a^2} \quad \text{or} \quad P(|X-\mu_X|\geq a) \leq \frac{\sigma_X^2}{a^2}$$

Note that by setting $a = k\sigma_X$ in Eq. (2.97), we obtain

$$P(|X-\mu_X|\geq k\sigma_X) \leq \frac{1}{k^2} \tag{2.98}$$

Equation (2.98) says that the probability that a r.v. will fall $k$ or more standard deviations from its mean is $\leq 1/k^2$. Notice that nothing at all is said about the distribution function of $X$. The Chebyshev inequality is therefore quite a generalized statement. However, when applied to a particular case, it may be quite weak.

## SPECIAL DISTRIBUTIONS

**2.37.** A binary source generates digits 1 and 0 randomly with probabilities 0.6 and 0.4, respectively.

(*a*) What is the probability that two 1s and three 0s will occur in a five-digit sequence?

(*b*) What is the probability that at least three 1s will occur in a five-digit sequence?

(*a*) Let $X$ be the r.v. denoting the number of 1s generated in a five-digit sequence. Since there are only two possible outcomes (1 or 0), the probability of generating 1 is constant, and there are five digits, it is clear that $X$ is a binomial r.v. with parameters $(n, p) = (5, 0.6)$. Hence, by Eq. (2.36), the probability that two 1s and three 0s will occur in a five-digit sequence is

$$P(X=2) = \binom{5}{2}(0.6)^2(0.4)^3 = 0.23$$

(*b*) The probability that at least three 1s will occur in a five-digit sequence is

$$P(X\geq 3) = 1 - P(X\leq 2)$$

where $$P(X\leq 2) = \sum_{k=0}^{2}\binom{5}{k}(0.6)^k(0.4)^{5-k} = 0.317$$

Hence, $$P(X\geq 3) = 1 - 0.317 = 0.683$$

**2.38.** A fair coin is flipped 10 times. Find the probability of the occurrence of 5 or 6 heads.

Let the r.v. $X$ denote the number of heads occurring when a fair coin is flipped 10 times. Then $X$ is a binomial r.v. with parameters $(n, p) = (10, \frac{1}{2})$. Thus, by Eq. (2.36),

$$P(5\leq X\leq 6) = \sum_{k=5}^{6}\binom{10}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{10-k} = 0.451$$

**2.39.** Let $X$ be a binomial r.v. with parameters $(n, p)$, where $0 < p < 1$. Show that as $k$ goes from 0 to $n$, the pmf $p_X(k)$ of $X$ first increases monotonically and then decreases monotonically, reaching its largest value when $k$ is the largest integer less than or equal to $(n+1)p$.

By Eq. (2.36), we have

$$\frac{p_X(k)}{p_X(k-1)} = \frac{\binom{n}{k}p^k(1-p)^{n-k}}{\binom{n}{k-1}p^{k-1}(1-p)^{n-k+1}} = \frac{(n-k+1)p}{k(1-p)} \tag{2.99}$$

Hence, $p_X(k) \geq p_X(k-1)$ if and only if $(n-k+1)p \geq k(1-p)$ or $k \leq (n+1)p$. Thus, we see that $p_X(k)$ increases monotonically and reaches its maximum when $k$ is the largest integer less than or equal to $(n+1)p$ and then decreases monotonically.

**2.40.** Show that the Poisson distribution can be used as a convenient approximation to the binomial distribution for large $n$ and small $p$.

From Eq. (2.36), the pmf of the binomial r.v. with parameters $(n, p)$ is

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} p^k (1-p)^{n-k}$$

Multiplying and dividing the right-hand side by $n^k$, we have

$$\binom{n}{k} p^k (1-p)^{n-k} = \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)}{k!} (np)^k \left(1-\frac{np}{n}\right)^{n-k}$$

If we let $n \to \infty$ in such a way that $np = \lambda$ remains constant, then

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \xrightarrow[n\to\infty]{} 1$$

$$\left(1-\frac{np}{n}\right)^{n-k} = \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-k} \xrightarrow[n\to\infty]{} e^{-\lambda}(1) = e^{-\lambda}$$

where we used the fact that

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$$

Hence, in the limit as $n \to \infty$ with $np = \lambda$ (and as $p = \lambda/n \to 0$),

$$\binom{n}{k} p^k (1-p)^{n-k} \xrightarrow[n\to\infty]{} e^{-\lambda} \frac{\lambda^k}{k!} \qquad np = \lambda$$

Thus, in the case of large $n$ and small $p$,

$$\binom{n}{k} p^k (1-p)^{n-k} \approx e^{-\lambda} \frac{\lambda^k}{k!} \qquad np = \lambda \tag{2.100}$$

which indicates that the binomial distribution can be approximated by the Poisson distribution.

**2.41.** A noisy transmission channel has a per-digit error probability $p = 0.01$.

(*a*) Calculate the probability of more than one error in 10 received digits.

(*b*) Repeat (*a*), using the Poisson approximation Eq. (2.100).

(*a*) It is clear that the number of errors in 10 received digits is a binomial r.v. $X$ with parameters $(n, p) = (10, 0.01)$. Then, using Eq. (2.36), we obtain

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \\ &= 0.0042 \end{aligned}$$

(*b*) Using Eq. (2.100) with $\lambda = np = 10(0.01) = 0.1$, we have

$$\begin{aligned} P(X > 1) &= 1 - P(X = 0) - P(X = 1) \\ &= 1 - e^{-0.1}\frac{(0.1)^0}{0!} - e^{-0.1}\frac{(0.1)^1}{1!} \\ &= 0.0047 \end{aligned}$$

**2.42.** The number of telephone calls arriving at a switchboard during any 10-minute period is known to be a Poisson r.v. $X$ with $\lambda = 2$.

(*a*) Find the probability that more than three calls will arrive during any 10-minute period.

(*b*) Find the probability that no calls will arrive during any 10-minute period.

(*a*) From Eq. (*2.40*), the pmf of $X$ is

$$p_X(k) = P(X = k) = e^{-2}\frac{2^k}{k!} \qquad k = 0, 1, \ldots$$

Thus,
$$P(X > 3) = -P(X \le 3) = 1 - \sum_{k=0}^{3} e^{-2}\frac{2^k}{k!}$$
$$= 1 - e^{-2}(1 + 2 + \tfrac{4}{2} + \tfrac{8}{6}) \approx 0.143$$

(*b*) $P(X = 0) = p_X(0) = e^{-2} \approx 0.135$

**2.43.** Consider the experiment of throwing a pair of fair dice.

(*a*) Find the probability that it will take less than six tosses to throw a 7.

(*b*) Find the probability that it will take more than six tosses to throw a 7.

(*a*) From Prob. 1.31(*a*), we see that the probability of throwing a 7 on any toss is $\frac{1}{6}$. Let $X$ denote the number of tosses required for the first success of throwing a 7. Then, from Prob. 2.15, it is clear that $X$ is a geometric r.v. with parameter $p = \frac{1}{6}$. Thus, using Eq. (*2.71*) of Prob. 2.15, we obtain

$$P(X < 6) = P(X \le 5) = F_X(5) = 1 - (\tfrac{5}{6})^5 \approx 0.598$$

(*b*) Similarly, we get

$$P(X > 6) = 1 - P(X \le 6) = 1 - F_X(6)$$
$$= 1 - [1 - (\tfrac{5}{6})^6] = (\tfrac{5}{6})^6 \approx 0.335$$

**2.44.** Consider the experiment of rolling a fair die. Find the average number of rolls required in order to obtain a 6.

Let $X$ denote the number of trials (rolls) required until the number 6 first appears. Then $X$ is a geometrical r.v. with parameter $p = \frac{1}{6}$. From Eq. (*2.85*) of Prob. 2.27, the mean of $X$ is given by

$$\mu_X = E(X) = \frac{1}{p} = \frac{1}{\frac{1}{6}} = 6$$

Thus, the average number of rolls required in order to obtain a 6 is 6.

**2.45.** Assume that the length of a phone call in minutes is an exponential r.v. $X$ with parameter $\lambda = \frac{1}{10}$. If someone arrives at a phone booth just before you arrive, find the probability that you will have to wait (*a*) less than 5 minutes, and (*b*) between 5 and 10 minutes.

(*a*) From Eq. (*2.48*), the pdf of $X$ is

$$f_X(x) = \begin{cases} \frac{1}{10}e^{-x/10} & x > 0 \\ 0 & x < 0 \end{cases}$$

Then

$$P(X < 5) = \int_0^5 \tfrac{1}{10}e^{-x/10}\,dx = -e^{-x/10}\Big|_0^5 = 1 - e^{-0.5} \approx 0.393$$

(*b*) Similarly,

$$P(5 < X < 10) = \int_5^{10} \tfrac{1}{10}e^{-x/10}\,dx = e^{-0.5} - e^{-1} \approx 0.239$$

**2.46.** All manufactured devices and machines fail to work sooner or later. Suppose that the failure rate is constant and the time to failure (in hours) is an exponential r.v. $X$ with parameter $\lambda$.

(*a*) Measurements show that the probability that the time to failure for computer memory chips in a given class exceeds $10^4$ hours is $e^{-1}$ ($\approx 0.368$). Calculate the value of the parameter $\lambda$.

(*b*) Using the value of the parameter $\lambda$ determined in part (*a*), calculate the time $x_0$ such that the probability that the time to failure is less than $x_0$ is 0.05.

(*a*) From Eq. (*2.49*), the cdf of $X$ is given by

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Now
$$P(X > 10^4) = 1 - P(X \le 10^4) = 1 - F_X(10^4)$$
$$= 1 - (1 - e^{-\lambda 10^4}) = e^{-\lambda 10^4} = e^{-1}$$

from which we obtain $\lambda = 10^{-4}$.

(*b*) We want

$$F_X(x_0) = P(X \le x_0) = 0.05$$

Hence,
$$1 - e^{-\lambda x_0} = 1 - e^{-10^{-4} x_0} = 0.05$$

or
$$e^{-10^{-4} x_0} = 0.95$$

from which we obtain

$$x_0 = -10^4 \ln(0.95) = 513 \text{ hours}$$

**2.47.** A production line manufactures 1000-ohm ($\Omega$) resistors that have 10 percent tolerance. Let $X$ denote the resistance of a resistor. Assuming that $X$ is a normal r.v. with mean 1000 and variance 2500, find the probability that a resistor picked at random will be rejected.

Let $A$ be the event that a resistor is rejected. Then $A = \{X < 900\} \cup \{X > 1100\}$. Since $\{X < 900\} \cap \{X > 1100\} = \varnothing$, we have

$$P(A) = P(X < 900) + P(X > 1100) = F_X(900) + [1 - F_X(1100)]$$

Since $X$ is a normal r.v. with $\mu = 1000$ and $\sigma^2 = 2500$ ($\sigma = 50$), by Eq. (*2.55*) and Table A (Appendix A),

$$F_X(900) = \Phi\left(\frac{900 - 1000}{50}\right) = \Phi(-2) = 1 - \Phi(2)$$

$$F_X(1100) = \Phi\left(\frac{1100 - 1000}{50}\right) = \Phi(2)$$

Thus,
$$P(A) = 2[1 - \Phi(2)] \approx 0.045$$

**2.48.** The radial miss distance [in meters (m)] of the landing point of a parachuting sky diver from the center of the target area is known to be a Rayleigh r.v. $X$ with parameter $\sigma^2 = 100$.

(*a*) Find the probability that the sky diver will land within a radius of 10 m from the center of the target area.

(*b*) Find the radius $r$ such that the probability that $X > r$ is $e^{-1}$ ($\approx 0.368$).

(*a*) Using Eq. (*2.75*) of Prob. 2.23, we obtain

$$P(X \le 10) = F_X(10) = 1 - e^{-100/200} = 1 - e^{-0.5} \approx 0.393$$

(*b*) Now

$$P(X > r) = 1 - P(X \le r) = 1 - F_X(r)$$
$$= 1 - (1 - e^{-r^2/200}) = e^{-r^2/200} = e^{-1}$$

from which we obtain $r^2 = 200$ and $r = \sqrt{200} = 14.142$ m.

## CONDITIONAL DISTRIBUTIONS

**2.49.** Let $X$ be a Poisson r.v. with parameter $\lambda$. Find the conditional pmf of $X$ given $B = (X \text{ is even})$.

From Eq. (*2.40*), the pdf of $X$ is

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad k = 0, 1, \ldots$$

Then the probability of event $B$ is

$$P(B) = P(X = 0, 2, 4, \ldots) = \sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

Let $A = \{X \text{ is odd}\}$. Then the probability of event $A$ is

$$P(A) = P(X = 1, 3, 5, \ldots) = \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}$$

Now

$$\sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} + \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}e^{\lambda} = 1 \tag{2.101}$$

$$\sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} - \sum_{k=\text{odd}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(-\lambda)^k}{k!} = e^{-\lambda}e^{-\lambda} = e^{-2\lambda} \tag{2.102}$$

Hence, adding Eqs. (*2.101*) and (*2.102*), we obtain

$$P(B) = \sum_{k=\text{even}}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = \tfrac{1}{2}(1 + e^{-2\lambda}) \tag{2.103}$$

Now, by Eq. (*2.62*), the pmf of $X$ given $B$ is

$$p_X(k \mid B) = \frac{P\{(X = k) \cap B\}}{P(B)}$$

If $k$ is even, $(X = k) \subset B$ and $(X = k) \cap B = (X = k)$. If $k$ is odd, $(X = k) \cap B = \varnothing$. Hence,

$$p_X(k \mid B) = \begin{cases} \dfrac{P(X = k)}{P(B)} = \dfrac{2e^{-\lambda}\lambda^k}{(1 + e^{-2\lambda})k!} & k \text{ even} \\ \dfrac{P(\varnothing)}{P(B)} = 0 & k \text{ odd} \end{cases}$$

**2.50.** Show that the conditional cdf and pdf of $X$ given the event $B = (a < X \le b)$ are as follows:

$$F_X(x \mid a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a < x \le b \\ 1 & x > b \end{cases} \tag{2.104}$$

$$f_X(x \mid a < X \le b) = \begin{cases} 0 & x \le a \\ \dfrac{f_X(x)}{\displaystyle\int_a^b f_X(\xi)\, d\xi} & a < x \le b \\ 0 & x > b \end{cases} \tag{2.105}$$

Substituting $B = (a < X \le b)$ in Eq. (2.59), we have

$$F_X(x \mid a < X \le b) = P(X \le x \mid a < X \le b) = \frac{P\{(X \le x) \cap (a < X \le b)\}}{P(a < X \le b)}$$

Now
$$(X \le x) \cap (a < X \le b) = \begin{cases} \varnothing & x \le a \\ (a < X \le x) & a < x \le b \\ (a < X \le b) & x > b \end{cases}$$

Hence,
$$F_X(x \mid a < X \le b) = \frac{P(\varnothing)}{P(a < X \le b)} = 0 \qquad x \le a$$

$$F_X(x \mid a < X \le b) = \frac{P(a < X \le x)}{P(a < X \le b)} = \frac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} \qquad a < x \le b$$

$$F_X(x \mid a < X \le b) = \frac{P(a < X \le b)}{P(a < X \le b)} = 1 \qquad x > b$$

By Eq. (2.63), the conditional pdf of $X$ given $a < X \le b$ is obtained by differentiating Eq. (2.104) with respect to $x$. Thus,

$$F_X(x \mid a < X \le b) = \begin{cases} 0 & X \le a \\ \dfrac{f_X(x)}{F_X(b) - F_X(a)} = \dfrac{f_X(x)}{\displaystyle\int_a^b f_X(\xi)\, d\xi} & a < x \le b \\ 0 & x > b \end{cases}$$

**2.51.** Recall the parachuting sky diver problem (Prob. 2.48). Find the probability of the sky diver landing within a 10-m radius from the center of the target area given that the landing is within 50 m from the center of the target area.

From Eq. (2.75) (Prob. 2.23) with $\sigma^2 = 100$, we have

$$F_X(x) = 1 - e^{-x^2/200}$$

Setting $x = 10$ and $b = 50$ and $a = -\infty$ in Eq. (2.104), we obtain

$$P(X \le 10 \mid X \le 50) = F_X(10 \mid X \le 50) = \frac{F_X(10)}{F_X(50)}$$
$$= \frac{1 - e^{-100/200}}{1 - e^{-2500/200}} \approx 0.393$$

**2.52.** Let $X = N(0; \sigma^2)$. Find $E(X \mid X > 0)$ and $\mathrm{Var}(X \mid X > 0)$.

From Eq. (2.52), the pdf of $X = N(0; \sigma^2)$ is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/(2\sigma^2)}$$

Then by Eq. (2.105),

$$f_X(x \mid X > 0) = \begin{cases} 0 & x < 0 \\ \dfrac{f_X(x)}{\displaystyle\int_0^\infty f_X(\xi)\, d\xi} = 2\dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/(2\sigma^2)} & x \ge 0 \end{cases} \tag{2.106}$$

Hence,
$$E(X \mid X > 0) = 2\frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty x e^{-x^2/(2\sigma^2)}\, dx$$

Let $y = x^2/(2\sigma^2)$. Then $dy = x\,dx/\sigma^2$, and we get

$$E(X \mid X > 0) = \frac{2\sigma}{\sqrt{2\pi}} \int_0^\infty e^{-y}\,dy = \sigma\sqrt{\frac{2}{\pi}} \tag{2.107}$$

Next,

$$E(X^2 \mid X > 0) = 2\,\frac{1}{\sqrt{2\pi}\,\sigma} \int_0^\infty x^2 e^{-x^2/(2\sigma^2)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^\infty x^2 e^{-x^2/(2\sigma^2)}\,dx = \mathrm{Var}(X) = \sigma^2 \tag{2.108}$$

Then by Eq. (*2.31*), we obtain

$$\mathrm{Var}(X \mid X > 0) = E(X^2 \mid X > 0) - [E(X \mid X > 0)]^2$$

$$= \sigma^2\left(1 - \frac{2}{\pi}\right) \approx 0.363\,\sigma^2 \tag{2.109}$$

**2.53.** A r.v. $X$ is said to be *without memory*, or *memoryless*, if

$$P(X \le x + t \mid X > t) = P(X \le x) \qquad x, t > 0 \tag{2.110}$$

Show that if $X$ is a nonnegative continuous r.v. which is memoryless, then $X$ must be an exponential r.v.

By Eq. (*1.39*), the memoryless condition (*2.110*) is equivalent to

$$\frac{P(X \le x + t,\ X > t)}{P(X > t)} = P(X \le x)$$

or

$$P(X \le x + t,\ X > t) = P(X \le x)P(X > t) \tag{2.111}$$

If $X$ is a nonnegative continuous r.v., then Eq. (*2.111*) becomes

$$P(t < X \le x + t) = P(0 < X \le x)P(X > t)$$

or [by Eq. (*2.25*)],

$$F_X(x + t) - F_X(t) = [F_X(x) - F_X(0)][1 - F_X(t)]$$

Noting that $F_X(0) = 0$ and rearranging the above equation, we get

$$\frac{F_X(x + t) - F_X(x)}{t} = \frac{F_X(t)[1 - F_X(x)]}{t}$$

Taking the limit as $t \to 0$, we obtain

$$F'_X(x) = F'_X(0)[1 - F_X(x)] \tag{2.112}$$

where $F'_X(x)$ denotes the derivative of $F_X(x)$. Let

$$R_X(x) = 1 - F_X(x) \tag{2.113}$$

Then Eq. (*2.112*) becomes

$$R'_X(x) = R'_X(0)R_X(x)$$

The solution to this differential equation is given by

$$R_X(x) = ke^{R'_X(0)x}$$

where $k$ is an integration constant. Noting that $k = R_X(0) = 1$ and letting $R'_X(0) = -F'_X(0) = -f_X(0) = -\lambda$, we obtain

$$R_X(x) = e^{-\lambda x}$$

and hence by Eq. (*2.113*),

$$F_X(x) = 1 - R_X(x) = 1 - e^{-\lambda x} \qquad x > 0$$

Thus, by Eq. (*2.49*), we conclude that $X$ is an exponential r.v. with parameter $\lambda = f_X(0)\,(>0)$.

Note that the memoryless property Eq. (*2.110*) is also known as the *Markov property* (see Chap. 5), and it may be equivalently expressed as

$$P(X > x + t \mid X > t) = P(X > x) \qquad x > 0,\ t > 0 \tag{2.114}$$

or

$$P(X > x + t) = P(X > x)P(X > t) \qquad x > 0,\ t > 0 \tag{2.115}$$

Let $X$ be the lifetime (in hours) of a component. Then Eq. (*2.114*) states that the probability that the component will operate for at least $x + t$ hours given that it has been operational for $t$ hours is the same as the initial probability that it will operate for at least $x$ hours. In other words, the component "forgets" how long it has been operating.

Note that Eq. (*2.115*) is satisfied when $X$ is an exponential r.v., since $P(X > x) = 1 - F_X(x) = e^{-\lambda x}$ and $e^{-\lambda(x+t)} = e^{-\lambda x}e^{-\lambda t}$.

## Supplementary Problems

**2.54.** Consider the experiment of tossing a coin. Heads appear about once out of every three tosses. If this experiment is repeated, what is the probability of the event that heads appear exactly twice during the first five tosses?

*Ans.* 0.329

**2.55.** Consider the experiment of tossing a fair coin three times (Prob. 1.1). Let $X$ be the r.v. that counts the number of heads in each sample point. Find the following probabilities:
(*a*) $P(X \leq 1)$; (*b*) $P(X > 1)$; and (*c*) $P(0 < X < 3)$.

*Ans.* (*a*) $\frac{1}{2}$, (*b*) $\frac{1}{2}$, (*c*) $\frac{3}{4}$

**2.56.** Consider the experiment of throwing two fair dice (Prob. 1.31). Let $X$ be the r.v. indicating the sum of the numbers that appear.

(*a*) What is the range of $X$?

(*b*) Find (i) $P(X = 3)$; (ii) $P(X \leq 4)$; and (iii) $P(3 < X \leq 7)$.

*Ans.* (*a*) $R_X = \{2, 3, 4, \ldots, 12\}$
(*b*) (i) $\frac{1}{18}$; (ii) $\frac{1}{6}$; (iii) $\frac{1}{2}$

**2.57.** Let $X$ denote the number of heads obtained in the flipping of a fair coin twice.

(*a*) Find the pmf of $X$.

(*b*) Compute the mean and the variance of $X$.

*Ans.* (*a*) $p_X(0) = \frac{1}{4}$, $p_X(1) = \frac{1}{2}$, $p_X(2) = \frac{1}{4}$
(*b*) $E(X) = 1$, $\mathrm{Var}(X) = \frac{1}{2}$

**2.58.** Consider the discrete r.v. $X$ that has the pmf

$$p_X(x_k) = (\tfrac{1}{2})^{x_k} \qquad x_k = 1, 2, 3, \ldots$$

Let $A = \{\zeta\colon X(\zeta) = 1, 3, 5, 7, \ldots\}$. Find $P(A)$.

*Ans.* $\frac{2}{3}$

**2.59.** Consider the function given by

$$p(x) = \begin{cases} \dfrac{k}{x^2} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant. Find the value of $k$ such that $p(x)$ can be the pmf of a discrete r.v. $X$.

*Ans.* $k = 6/\pi^2$

**2.60.** It is known that the floppy disks produced by company $A$ will be defective with probability 0.01. The company sells the disks in packages of 10 and offers a guarantee of replacement that at most 1 of the 10 disks is defective. Find the probability that a package purchased will have to be replaced.

*Ans.* 0.004

**2.61.** Given that $X$ is a Poisson r.v. and $p_X(0) = 0.0498$, compute $E(X)$ and $P(X \geq 3)$.

*Ans.* $E(X) = 3$, $P(X \geq 3) = 0.5767$

**2.62.** A digital transmission system has an error probability of $10^{-6}$ per digit. Find the probability of three or more errors in $10^6$ digits by using the Poisson distribution approximation.

*Ans.* 0.08

**2.63.** Show that the pmf $p_X(x)$ of a Poisson r.v. $X$ with parameter $\lambda$ satisfies the following recursion formula:

$$p_X(k+1) = \frac{\lambda}{k+1} p_X(k) \qquad p_X(k-1) = \frac{k}{\lambda} p_X(k)$$

*Hint:* Use Eq. (*2.40*).

**2.64.** The continuous r.v. $X$ has the pdf

$$f_X(x) = \begin{cases} k(x - x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant. Find the value of $k$ and the cdf of $X$.

*Ans.* $k = 6$; $F_X(x) = \begin{cases} 0 & x \leq 0 \\ 3x^2 - 2x^3 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$

**2.65.** The continuous r.v. $X$ has the pdf

$$f_X(x) = \begin{cases} k(2x - x^2) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant. Find the value of $k$ and $P(X > 1)$.

*Ans.* $k = \frac{3}{4}$; $P(X > 1) = \frac{1}{2}$

**2.66.** A r.v. $X$ is defined by the cdf

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}x & 0 \leq x < 1 \\ k & 1 \leq x \end{cases}$$

(*a*) Find the value of $k$.

(*b*) Find the type of $X$.

(*c*) Find (i) $P(\frac{1}{2} < X \leq 1)$; (ii) $P(\frac{1}{2} < X < 1)$; and (iii) $P(X > 2)$.

*Ans.* (*a*) $k = 1$.

(*b*) Mixed r.v.

(*c*) (i) $\frac{3}{4}$; (ii) $\frac{1}{4}$; (iii) 0

**2.67.** It is known that the time (in hours) between consecutive traffic accidents can be described by the exponential r.v. $X$ with parameter $\lambda = \frac{1}{60}$. Find (i) $P(X \leq 60)$; (ii) $P(X > 120)$; and (iii) $P(10 < X \leq 100)$.

*Ans.* (i) 0.632; (ii) 0.135; (iii) 0.658

**2.68.** Binary data are transmitted over a noisy communication channel in block of 16 binary digits. The probability that a received digit is in error as a result of channel noise is 0.01. Assume that the errors occurring in various digit positions within a block are independent.

(*a*) Find the mean and the variance of the number of errors per block.

(*b*) Find the probability that the number of errors per block is greater than or equal to 4.

*Ans.* (*a*) $E(X) = 0.16$, $\text{Var}(X) = 0.158$

(*b*) $0.165 \times 10^{-4}$

**2.69.** Let the continuous r.v. $X$ denote the weight (in pounds) of a package. The range of weight of packages is between 45 and 60 pounds.

(*a*) Determine the probability that a package weighs more than 50 pounds.

(*b*) Find the mean and the variance of the weight of packages.

*Hint:* Assume that $X$ is uniformly distributed over (45, 60).

*Ans.* (*a*) $\frac{2}{3}$; (*b*) $E(X) = 52.5$, $\text{Var}(X) = 18.75$

**2.70.** In the manufacturing of computer memory chips, company A produces one defective chip for every nine good chips. Let $X$ be time to failure (in months) of chips. It is known that $X$ is an exponential r.v. with parameter $\lambda = \frac{1}{2}$ for a defective chip and $\lambda = \frac{1}{10}$ with a good chip. Find the probability that a chip purchased randomly will fail before (*a*) six months of use; and (*b*) one year of use.

*Ans.* (*a*) 0.501; (*b*) 0.729

**2.71.** The *median* of a continuous r.v. $X$ is the value of $x = x_0$ such that $P(X \geq x_0) = P(X \leq x_0)$. The *mode* of $X$ is the value of $x = x_m$ at which the pdf of $X$ achieves its maximum value.

(*a*) Find the median and mode of an exponential r.v. $X$ with parameter $\lambda$.

(*b*) Find the median and mode of a normal r.v. $X = N(\mu, \sigma^2)$.

*Ans.* (*a*) $x_0 = (\ln 2)/\lambda = 0.693/\lambda$, $x_m = 0$

(*b*) $x_0 = x_m = \mu$

**2.72.** Let the r.v. $X$ denote the number of defective components in a random sample of $n$ components, chosen without replacement from a total of $N$ components, $r$ of which are defective. The r.v. $X$ is known as the *hypergeometric* r.v. with parameters $(N, r, n)$.

(*a*) Find the pmf of $X$.

(*b*) Find the mean and variance of $X$.

*Hint:* To find $E(X)$, note that

$$\binom{r}{x} = \frac{r}{x}\binom{r-1}{x-1} \quad \text{and} \quad \binom{N}{n} = \sum_{x=0}^{n} \binom{r}{x}\binom{N-r}{n-x}$$

To find $\text{Var}(X)$, first find $E[X(X-1)]$.

*Ans.* (*a*) $p_X(x) = \dfrac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$ $\quad x = 0, 1, 2, \ldots, \min\{r, n\}$

(*b*) $E(X) = n\left(\dfrac{r}{N}\right)$, $\mathrm{Var}(X) = n\left(\dfrac{r}{N}\right)\left(1 - \dfrac{r}{N}\right)\left(\dfrac{N-n}{N-1}\right)$

**2.73.** A lot consisting of 100 fuses is inspected by the following procedure: Five fuses are selected randomly, and if all five "blow" at the specified amperage, the lot is accepted. Suppose that the lot contains 10 defective fuses. Find the probability of accepting the lot.

*Hint:* Let $X$ be a r.v. equal to the number of defective fuses in the sample of 5 and use the result of Prob. 2.72.

*Ans.* 0.584

**2.74.** Consider the experiment of observing a sequence of Bernoulli trials until exactly $r$ successes occur. Let the r.v. $X$ denote the number of trials needed to observe the $r$th success. The r.v. $X$ is known as the *negative binomial* r.v. with parameter $p$, where $p$ is the probability of a success at each trial.

(*a*) Find the pmf of $X$.

(*b*) Find the mean and variance of $X$.

*Hint:* To find $E(X)$, use Maclaurin's series expansions of the negative binomial $h(q) = (1-q)^{-r}$ and its derivatives $h'(q)$ and $h''(q)$, and note that

$$h(q) = (1-q)^{-r} = \sum_{k=0}^{\infty} \binom{r+k-1}{r-1} q^k = \sum_{x=r}^{\infty} \binom{x-1}{r-1} q^{x-r}$$

To find $\mathrm{Var}(X)$, first find $E[(X-r)(X-r-1)]$ using $h''(q)$.

*Ans.* (*a*) $p_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$ $\quad x = r, r+1, \ldots$

(*b*) $E(X) = r\left(\dfrac{1}{p}\right)$, $\mathrm{Var}(X) = \dfrac{r(1-p)}{p^2}$

**2.75.** Suppose the probability that a bit transmitted through a digital communication channel and received in error is 0.1. Assuming that the transmissions are independent events, find the probability that the third error occurs at the 10th bit.

*Ans.* 0.017

**2.76.** A r.v. $X$ is called a *Laplace* r.v. if its pdf is given by

$$f_X(x) = ke^{-\lambda|x|} \qquad \lambda > 0, \quad -\infty < x < \infty$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the cdf of $X$.

(*c*) Find the mean and the variance of $X$.

*Ans.* (*a*) $k = \lambda/2$ (*b*) $F_X(x) = \begin{cases} \frac{1}{2}e^{\lambda x} & x < 0 \\ 1 - \frac{1}{2}e^{-\lambda x} & x \geq 0 \end{cases}$

(*c*) $E(X) = 0$, $\mathrm{Var}(X) = 2/\lambda^2$

**2.77.** A r.v. $X$ is called a *Cauchy* r.v. if its pdf is given by

$$f_X(x) = \frac{k}{a^2 + x^2} \qquad -\infty < x < \infty$$

where $a\,(>0)$ and $k$ are constants.

(*a*) Find the value of $k$.

(*b*) Find the cdf of $X$.

(*c*) Find the mean and the variance of $X$.

*Ans.* (*a*) $k = a/\pi$ (*b*) $F_X(x) = \dfrac{1}{2} + \dfrac{1}{\pi}\tan^{-1}\left(\dfrac{x}{a}\right)$

(*c*) $E(X) = 0$, $\mathrm{Var}(X)$ does not exist.

# Chapter 3

# Multiple Random Variables

## 3.1 INTRODUCTION

In many applications it is important to study two or more r.v.'s defined on the same sample space. In this chapter, we first consider the case of two r.v.'s, their associated distribution, and some properties, such as independence of the r.v.'s. These concepts are then extended to the case of many r.v.'s defined on the same sample space.

## 3.2. BIVARIATE RANDOM VARIABLES

### A. Definition:

Let $S$ be the sample space of a random experiment. Let $X$ and $Y$ be two r.v.'s. Then the pair $(X, Y)$ is called a *bivariate* r.v. (or *two-dimensional random vector*) if each of $X$ and $Y$ associates a real number with every element of $S$. Thus, the bivariate r.v. $(X, Y)$ can be considered as a function that to each point $\zeta$ in $S$ assigns a point $(x, y)$ in the plane (Fig. 3-1). The range space of the bivariate r.v. $(X, Y)$ is denoted by $R_{xy}$ and defined by

$$R_{xy} = \{(x, y);\ \zeta \in S \text{ and } X(\zeta) = x,\ Y(\zeta) = y\}$$

If the r.v.'s $X$ and $Y$ are each, by themselves, discrete r.v.'s, then $(X, Y)$ is called a *discrete* bivariate r.v. Similarly, if $X$ and $Y$ are each, by themselves, continuous r.v.'s, then $(X, Y)$ is called a *continuous* bivariate r.v. If one of $X$ and $Y$ is discrete while the other is continuous, then $(X, Y)$ is called a *mixed* bivariate r.v.

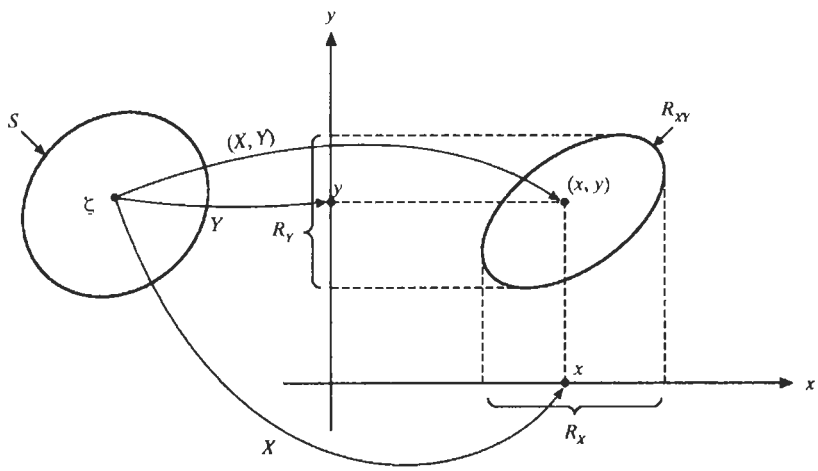



**Fig. 3-1** $(X, Y)$ as a function from $S$ to the plane.

## 3.3 JOINT DISTRIBUTION FUNCTIONS

### A. Definition:

The *joint cumulative distribution function* (or joint cdf) of $X$ and $Y$, denoted by $F_{XY}(x, y)$, is the function defined by

$$F_{XY}(x, y) = P(X \le x, Y \le y) \tag{3.1}$$

The event $(X \le x, Y \le y)$ in Eq. (3.1) is equivalent to the event $A \cap B$, where $A$ and $B$ are events of $S$ defined by

$$A = \{\zeta \in S;\ X(\zeta) \le x\} \qquad \text{and} \qquad B = \{\zeta \in S;\ Y(\zeta) \le y\} \tag{3.2}$$

and

$$P(A) = F_X(x) \qquad P(B) = F_Y(y)$$

Thus,

$$F_{XY}(x, y) = P(A \cap B) \tag{3.3}$$

If, for particular values of $x$ and $y$, $A$ and $B$ were independent events of $S$, then by Eq. (1.46),

$$F_{XY}(x, y) = P(A \cap B) = P(A)P(B) = F_X(x)F_Y(y)$$

### B. Independent Random Variables:

Two r.v.'s $X$ and $Y$ will be called *independent* if

$$F_{XY}(x, y) = F_X(x)F_Y(y) \tag{3.4}$$

for every value of $x$ and $y$.

### C. Properties of $F_{XY}(x, y)$:

The joint cdf of two r.v.'s has many properties analogous to those of the cdf of a single r.v.

1. $0 \le F_{XY}(x, y) \le 1$ (3.5)

2. If $x_1 \le x_2$, and $y_1 \le y_2$, then

$$F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_1) \le F_{XY}(x_2, y_2) \tag{3.6a}$$

$$F_{XY}(x_1, y_1) \le F_{XY}(x_1, y_2) \le F_{XY}(x_2, y_2) \tag{3.6b}$$

3. $\lim\limits_{\substack{x \to \infty \\ y \to \infty}} F_{XY}(x, y) = F_{XY}(\infty, \infty) = 1$ (3.7)

4. $\lim\limits_{x \to -\infty} F_{XY}(x, y) = F_{XY}(-\infty, y) = 0$ (3.8a)

$\lim\limits_{y \to -\infty} F_{XY}(x, y) = F_{XY}(x, -\infty) = 0$ (3.8b)

5. $\lim\limits_{x \to a^+} F_{XY}(x, y) = F_{XY}(a^+, y) = F_{XY}(a, y)$ (3.9a)

$\lim\limits_{y \to b^+} F_{XY}(x, y) = F_{XY}(x, b^+) = F_{XY}(x, b)$ (3.9b)

6. $P(x_1 < X \le x_2, Y \le y) = F_{XY}(x_2, y) - F_{XY}(x_1, y)$ (3.10)

$P(X \le x, y_1 < Y \le y_2) = F_{XY}(x, y_2) - F_{XY}(x, y_1)$ (3.11)

7. If $x_1 \le x_2$ and $y_1 \le y_2$, then

$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1) \ge 0 \tag{3.12}$$

Note that the left-hand side of Eq. (3.12) is equal to $P(x_1 < X \le x_2, y_1 < Y \le y_2)$ (Prob. 3.5).

**D. Marginal Distribution Functions:**

Now $$\lim_{y \to \infty} (X \leq x, Y \leq y) = (X \leq x, Y \leq \infty) = (X \leq x)$$

since the condition $y \leq \infty$ is always satisfied. Then

$$\lim_{y \to \infty} F_{XY}(x, y) = F_{XY}(x, \infty) = F_X(x) \tag{3.13}$$

Similarly, $$\lim_{x \to \infty} F_{XY}(x, y) = F_{XY}(\infty, y) = F_Y(y) \tag{3.14}$$

The cdf's $F_X(x)$ and $F_Y(y)$, when obtained by Eqs. (3.13) and (3.14), are referred to as the *marginal cdf's* of $X$ and $Y$, respectively.

## 3.4 DISCRETE RANDOM VARIABLES—JOINT PROBABILITY MASS FUNCTIONS

**A. Joint Probability Mass Functions:**

Let $(X, Y)$ be a discrete bivariate r.v., and let $(X, Y)$ take on the values $(x_i, y_j)$ for a certain allowable set of integers $i$ and $j$. Let

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) \tag{3.15}$$

The function $p_{XY}(x_i, y_j)$ is called the *joint probability mass function* (joint pmf) of $(X, Y)$.

**B. Properties of $p_{XY}(x_i, y_j)$:**

1. $0 \leq p_{XY}(x_i, y_j) \leq 1$ (3.16)
2. $\sum_{x_i} \sum_{y_j} p_{XY}(x_i, y_j) = 1$ (3.17)
3. $P[(X, Y) \in A] = \sum\sum_{(x_i, y_j) \in R_A} p_{XY}(x_i, y_j)$ (3.18)

where the summation is over the points $(x_i, y_j)$ in the range space $R_A$ corresponding to the event $A$. The joint cdf of a discrete bivariate r.v. $(X, Y)$ is given by

$$F_{XY}(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} p_{XY}(x_i, y_j) \tag{3.19}$$

**C. Marginal Probability Mass Functions:**

Suppose that for a fixed value $X = x_i$, the r.v. $Y$ can take on only the possible values $y_j$ ($j = 1, 2, \ldots, n$). Then

$$P(X = x_i) = p_X(x_i) = \sum_{y_j} p_{XY}(x_i, y_j) \tag{3.20}$$

where the summation is taken over all possible pairs $(x_i, y_j)$ with $x_i$ fixed. Similarly,

$$P(Y = y_j) = p_Y(y_j) = \sum_{x_i} p_{XY}(x_i, y_j) \tag{3.21}$$

where the summation is taken over all possible pairs $(x_i, y_j)$ with $y_j$ fixed. The pmf's $p_X(x_i)$ and $p_Y(y_j)$, when obtained by Eqs. (3.20) and (3.21), are referred to as the *marginal* pmf's of $X$ and $Y$, respectively.

**D. Independent Random Variables:**

If $X$ and $Y$ are independent r.v.'s, then (Prob. 3.10)

$$p_{XY}(x_i, y_j) = p_X(x_i) p_Y(y_j) \tag{3.22}$$

## 3.5 CONTINUOUS RANDOM VARIABLES—JOINT PROBABILITY DENSITY FUNCTIONS

### A. Joint Probability Density Functions:

Let $(X, Y)$ be a continuous bivariate r.v. with cdf $F_{XY}(x, y)$ and let

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \, \partial y} \tag{3.23}$$

The function $f_{XY}(x, y)$ is called the *joint probability density function* (joint pdf) of $(X, Y)$. By integrating Eq. (3.23), we have

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\xi, \eta) \, d\eta \, d\xi \tag{3.24}$$

### B. Properties of $f_{XY}(x, y)$:

1. $f_{XY}(x, y) \geq 0$ (3.25)
2. $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \, dy = 1$ (3.26)
3. $f_{XY}(x, y)$ is continuous for all values of $x$ or $y$ except possibly a finite set.
4. $P[(X, Y) \in A] = \displaystyle\iint_{R_A} f_{XY}(x, y) \, dx \, dy$ (3.27)
5. $P(a < X \leq b, c < Y \leq d) = \displaystyle\int_{c}^{d} \int_{a}^{b} f_{XY}(x, y) \, dx \, dy$ (3.28)

Since $P(X = a) = 0 = P(Y = c)$ [by Eq. (2.19)], it follows that

$$\begin{aligned} P(a < X \leq b, c < Y \leq d) &= P(a \leq X \leq b, c \leq Y \leq d) = P(a \leq X < b, c \leq Y < d) \\ &= P(a < X < b, c < Y < d) = \int_{c}^{d} \int_{a}^{b} f_{XY}(x, y) \, dx \, dy \end{aligned} \tag{3.29}$$

### C. Marginal Probability Density Functions:

By Eq. (3.13),

$$F_X(x) = F_{XY}(x, \infty) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{XY}(\xi, \eta) \, d\eta \, d\xi$$

Hence

$$f_X(x) = \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, \eta) \, d\eta$$

or

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dy \tag{3.30}$$

Similarly,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y) \, dx \tag{3.31}$$

The pdf's $f_X(x)$ and $f_Y(y)$, when obtained by Eqs. (3.30) and (3.31), are referred to as the *marginal* pdf's of $X$ and $Y$, respectively.

### D. Independent Random Variables:

If $X$ and $Y$ are independent r.v.'s, by Eq. (3.4),

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

Then

$$\frac{\partial^2 F_{XY}(x, y)}{\partial x\, \partial y} = \frac{\partial}{\partial x} F_X(x) \frac{\partial}{\partial y} F_Y(y)$$

or

$$f_{XY}(x, y) = f_X(x)f_Y(y) \tag{3.32}$$

analogous with Eq. (3.22) for the discrete case. Thus, we say that the continuous r.v.'s $X$ and $Y$ are independent r.v.'s if and only if Eq. (3.32) is satisfied.

## 3.6 CONDITIONAL DISTRIBUTIONS

### A. Conditional Probability Mass Functions:

If $(X, Y)$ is a discrete bivariate r.v. with joint pmf $p_{XY}(x_i, y_j)$, then the *conditional* pmf of $Y$, given that $X = x_i$, is defined by

$$p_{Y|X}(y_j \mid x_i) = \frac{p_{XY}(x_i, y_j)}{p_X(x_i)} \qquad p_X(x_i) > 0 \tag{3.33}$$

Similarly, we can define $p_{X|Y}(x_i \mid y_j)$ as

$$p_{X|Y}(x_i \mid y_j) = \frac{p_{XY}(x_i, y_j)}{p_Y(y_j)} \qquad p_Y(y_j) > 0 \tag{3.34}$$

### B. Properties of $p_{Y|X}(y_j \mid x_i)$:

1. $0 \le p_{Y|X}(y_j \mid x_i) \le 1$ (3.35)
2. $\sum_{y_j} p_{Y|X}(y_j \mid x_i) = 1$ (3.36)

Notice that if $X$ and $Y$ are independent, then by Eq. (3.22),

$$p_{Y|X}(y_j \mid x_i) = p_Y(y_j) \qquad \text{and} \qquad p_{X|Y}(x_i \mid y_j) = p_X(x_i) \tag{3.37}$$

### C. Conditional Probability Density Functions:

If $(X, Y)$ is a continuous bvivariate r.v. with joint pdf $f_{XY}(x, y)$, then the *conditional* pdf of Y, given that $X = x$, is defined by

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} \qquad f_X(x) > 0 \tag{3.38}$$

Similarly, we can define $f_{X|Y}(x \mid y)$ as

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad f_Y(y) > 0 \tag{3.39}$$

### D. Properties of $f_{Y|X}(y \mid x)$:

1. $f_{Y|X}(y \mid x) \ge 0$ (3.40)
2. $\int_{-\infty}^{\infty} f_{Y|X}(y \mid x)\, dy = 1$ (3.41)

As in the discrete case, if $X$ and $Y$ are independent, then by Eq. (3.32),

$$f_{Y|X}(y \mid x) = f_Y(y) \qquad \text{and} \qquad f_{X|Y}(x \mid y) = f_X(x) \tag{3.42}$$

## 3.7 COVARIANCE AND CORRELATION COEFFICIENT

The $(k, n)$th moment of a bivariate r.v. $(X, Y)$ is defined by

$$m_{kn} = E(X^k Y^n) = \begin{cases} \sum_{y_j} \sum_{x_i} x_i^k y_j^n p_{XY}(x_i, y_j) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^k y^n f_{XY}(x, y)\, dx\, dy & \text{(continuous case)} \end{cases} \tag{3.43}$$

If $n = 0$, we obtain the $k$th moment of $X$, and if $k = 0$, we obtain the $n$th moment of $Y$. Thus,

$$m_{10} = E(X) = \mu_X \qquad \text{and} \qquad m_{01} = E(Y) = \mu_Y \tag{3.44}$$

If $(X, Y)$ is a discrete bivariate r.v., then using Eqs. (3.43), (3.20), and (3.21), we obtain

$$\begin{aligned} \mu_X = E(X) &= \sum_{y_j} \sum_{x_i} x_i p_{XY}(x_i, y_j) \\ &= \sum_{x_i} x_i \left[ \sum_{y_j} p_{XY}(x_i, y_j) \right] = \sum_{x_i} x_i p_X(x_i) \end{aligned} \tag{3.45a}$$

$$\begin{aligned} \mu_Y = E(Y) &= \sum_{x_i} \sum_{y_j} y_j p_{XY}(x_i, y_j) \\ &= \sum_{y_j} y_j \left[ \sum_{x_i} p_{XY}(x_i, y_j) \right] = \sum_{y_j} y_j p_Y(y_j) \end{aligned} \tag{3.45b}$$

Similarly, we have

$$E(X^2) = \sum_{y_j} \sum_{x_i} x_i^2 p_{XY}(x_i, y_j) = \sum_{x_i} x_i^2 p_X(x_i) \tag{3.46a}$$

$$E(Y^2) = \sum_{y_j} \sum_{x_i} y_j^2 p_{XY}(x_i, y_j) = \sum_{y_j} y_j^2 p_Y(y_j) \tag{3.46b}$$

If $(X, Y)$ is a continuous bivariate r.v., then using Eqs. (3.43), (3.30), and (3.31), we obtain

$$\begin{aligned} \mu_X = E(X) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{XY}(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} x \left[ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right] dx = \int_{-\infty}^{\infty} x f_X(x)\, dx \end{aligned} \tag{3.47a}$$

$$\begin{aligned} \mu_Y = E(Y) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{XY}(x, y)\, dx\, dy \\ &= \int_{-\infty}^{\infty} y \left[ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right] dy = \int_{-\infty}^{\infty} y f_Y(y)\, dy \end{aligned} \tag{3.47b}$$

Similarly, we have

$$E(X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \tag{3.48a}$$

$$E(Y^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy \tag{3.48b}$$

The variances of $X$ and $Y$ are easily obtained by using Eq. (*2.31*). The (1, 1)th joint moment of $(X, Y)$,

$$m_{11} = E(XY) \tag{3.49}$$

is called the *correlation* of $X$ and $Y$. If $E(XY) = 0$, then we say that $X$ and $Y$ are *orthogonal*. The *covariance* of $X$ and $Y$, denoted by $\mathrm{Cov}(X, Y)$ or $\sigma_{XY}$, is defined by

$$\mathrm{Cov}(X, Y) = \sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] \tag{3.50}$$

Expanding Eq. (*3.50*), we obtain

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) \tag{3.51}$$

If $\mathrm{Cov}(X, Y) = 0$, then we say that $X$ and $Y$ are *uncorrelated*. From Eq. (*3.51*), we see that $X$ and $Y$ are uncorrelated if

$$E(XY) = E(X)E(Y) \tag{3.52}$$

Note that if $X$ and $Y$ are independent, then it can be shown that they are uncorrelated (Prob. 3.32), but the converse is not true in general; that is, the fact that $X$ and $Y$ are uncorrelated does not, in general, imply that they are independent (Probs. 3.33, 3.34, and 3.38). The *correlation coefficient*, denoted by $\rho(X, Y)$ or $\rho_{XY}$, is defined by

$$\rho(X, Y) = \rho_{XY} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{3.53}$$

It can be shown that (Prob. 3.36)

$$|\rho_{XY}| \le 1 \qquad \text{or} \qquad -1 \le \rho_{XY} \le 1 \tag{3.54}$$

Note that the correlation coefficient of $X$ and $Y$ is a measure of linear dependence between $X$ and $Y$ (see Prob. 4.40).

## 3.8 CONDITIONAL MEANS AND CONDITIONAL VARIANCES

If $(X, Y)$ is a discrete bivariate r.v. with joint pmf $p_{XY}(x_i, y_j)$, then the *conditional mean* (or *conditional expectation*) of $Y$, given that $X = x_i$, is defined by

$$\mu_{Y|x_i} = E(Y \mid x_i) = \sum_{y_j} y_j p_{Y|X}(y_j \mid x_i) \tag{3.55}$$

The *conditional variance* of $Y$, given that $X = x_i$, is defined by

$$\sigma^2_{Y|x_i} = \mathrm{Var}(Y \mid x_i) = E[(Y - \mu_{Y|x_i})^2 \mid x_i] = \sum_{y_j} (y_j - \mu_{Y|x_i})^2 p_{Y|X}(y_j \mid x_i) \tag{3.56}$$

which can be reduced to

$$\mathrm{Var}(Y \mid x_i) = E(Y^2 \mid x_i) - [E(Y \mid x_i)]^2 \tag{3.57}$$

The conditional mean of $X$, given that $Y = y_j$, and the conditional variance of $X$, given that $Y = y_j$, are given by similar expressions. Note that the conditional mean of $Y$, given that $X = x_i$, is a function of $x_i$ alone. Similarly, the conditional mean of $X$, given that $Y = y_j$, is a function of $y_j$ alone.

If $(X, Y)$ is a continuous bivariate r.v. with joint pdf $f_{XY}(x, y)$, the conditional mean of $Y$, given that $X = x$, is defined by

$$\mu_{Y|x} = E(Y \mid x) = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\, dy \tag{3.58}$$

The conditional variance of $Y$, given that $X = x$, is defined by

$$\sigma^2_{Y|x} = \mathrm{Var}(Y \mid x) = E[(Y - \mu_{Y|x})^2 \mid x] = \int_{-\infty}^{\infty} (y - \mu_{Y|x})^2 f_{Y|X}(y \mid x)\, dy \tag{3.59}$$

which can be reduced to

$$\mathrm{Var}(Y \mid x) = E(Y^2 \mid x) - [E(Y \mid x)]^2 \tag{3.60}$$

The conditional mean of $X$, given that $Y = y$, and the conditional variance of $X$, given that $Y = y$, are given by similar expressions. Note that the conditional mean of $Y$, given that $X = x$, is a function of $x$ alone. Similarly, the conditional mean of $X$, given that $Y = y$, is a function of $y$ alone (Prob. 3.40).

## 3.9 *N*-VARIATE RANDOM VARIABLES

In previous sections, the extension from one r.v. to two r.v.'s has been made. The concepts can be extended easily to any number of r.v.'s defined on the same sample space. In this section we briefly describe some of the extensions.

### A. Definitions:

Given an experiment, the $n$-tuple of r.v.'s $(X_1, X_2, \ldots, X_n)$ is called an *n-variate* r.v. (or *n-dimensional random vector*) if each $X_i$, $i = 1, 2, \ldots, n$, associates a real number with every sample point $\zeta \in S$. Thus, an $n$-variate r.v. is simply a rule associating an $n$-tuple of real numbers with every $\zeta \in S$.

Let $(X_1, \ldots, X_n)$ be an $n$-variate r.v. on $S$. Then its joint cdf is defined as

$$F_{X_1 \cdots X_n}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n) \tag{3.61}$$

Note that

$$F_{X_1 \cdots X_n}(\infty, \ldots, \infty) = 1 \tag{3.62}$$

The marginal joint cdf's are obtained by setting the appropriate $X_i$'s to $+\infty$ in Eq. (3.61). For example,

$$F_{X_1 \cdots X_{n-1}}(x_1, \ldots, x_{n-1}) = F_{X_1 \cdots X_{n-1} X_n}(x_1, \ldots, x_{n-1}, \infty) \tag{3.63}$$

$$F_{X_1 X_2}(x_1, x_2) = F_{X_1 X_2 X_3 \cdots X_n}(x_1, x_2, \infty, \ldots, \infty) \tag{3.64}$$

A discrete $n$-variate r.v. will be described by a joint pmf defined by

$$p_{X_1 \cdots X_n}(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n) \tag{3.65}$$

The probability of any $n$-dimensional event $A$ is found by summing Eq. (3.65) over the points in the $n$-dimensional range space $R_A$ corresponding to the event $A$:

$$P[(X_1, \ldots, X_n) \in A] = \sum_{(x_1, \ldots, x_n) \in R_A} \cdots \sum p_{X_1 \cdots X_n}(x_1, \ldots, x_n) \tag{3.66}$$

***Properties of*** $p_{X_1 \cdots X_n}(x_1, \ldots, x_n)$**:**

1. $0 \le p_{X_1 \cdots X_n}(x_1, \ldots, x_n) \le 1$ (3.67)
2. $\sum_{x_1} \cdots \sum_{x_n} p_{X_1 \cdots X_n}(x_1, \ldots, x_n) = 1$ (3.68)

The marginal pmf's of one or more of the r.v.'s are obtained by summing Eq. (3.65) appropriately. For example,

$$p_{X_1 \cdots X_{n-1}}(x_1, \ldots, x_{n-1}) = \sum_{x_n} p_{X_1 \cdots X_n}(x_1, \ldots, x_n) \tag{3.69}$$

$$p_{X_1}(x_1) = \sum_{x_2} \cdots \sum_{x_n} p_{X_1 \cdots X_n}(x_1, \ldots, x_n) \tag{3.70}$$

Conditional pmf's are defined similarly. For example,

$$p_{X_n|X_1 \cdots X_{n-1}}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{p_{X_1 \cdots X_n}(x_1, \ldots, x_n)}{p_{X_1 \cdots X_{n-1}}(x_1, \ldots, x_{n-1})} \tag{3.71}$$

A continuous $n$-variate r.v. will be described by a joint pdf defined by

$$f_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \frac{\partial^n F_{X_1 \cdots X_n}(x_1, \ldots, x_n)}{\partial x_1 \cdots \partial x_n} \tag{3.72}$$

Then

$$F_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{X_1 \cdots X_n}(\xi_1, \ldots, \xi_n)\, d\xi_1 \cdots d\xi_n \tag{3.73}$$

and

$$P[(X_1, \ldots, X_n) \in A] = \int\cdots\int_{(x_1, \ldots, x_n) \in R_A} f_{X_1 \cdots X_n}(\xi_1, \ldots, \xi_n)\, d\xi_1 \cdots d\xi_n \tag{3.74}$$

***Properties of $f_{X_1 \cdots X_n}(x_1, \ldots, x_n)$:***

1. $f_{X_1 \cdots X_n}(x_1, \ldots, x_n) \geq 0$ (3.75)

2. $\displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n = 1$ (3.76)

The marginal pdf's of one or more of the r.v.'s are obtained by integrating Eq. (3.72) appropriately. For example,

$$f_{X_1 \cdots X_{n-1}}(x_1, \ldots, x_{n-1}) = \int_{-\infty}^{\infty} f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_n \tag{3.77}$$

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_2 \cdots dx_n \tag{3.78}$$

Conditional pdf's are defined similarly. For example,

$$f_{X_n|X_1 \cdots X_{n-1}}(x_n \mid x_1, \ldots, x_{n-1}) = \frac{f_{X_1 \cdots X_n}(x_1, \ldots, x_n)}{f_{X_1 \cdots X_{n-1}}(x_1, \ldots, x_{n-1})} \tag{3.79}$$

The r.v.'s $X_1, \ldots, X_n$ are said to be mutually independent if

$$p_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} p_{X_i}(x_i) \tag{3.80}$$

for the discrete case, and

$$f_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \tag{3.81}$$

for the continuous case.

The mean (or expectation) of $X_i$ in $(X_1, \ldots, X_n)$ is defined as

$$\mu_i = E(X_i) = \begin{cases} \displaystyle\sum_{x_n} \cdots \sum_{x_1} x_i\, p_{X_1 \cdots X_n}(x_1, \ldots, x_n) & \text{(discrete case)} \\ \displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} x_i\, f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \text{(continuous case)} \end{cases} \tag{3.82}$$

The variance of $X_i$ is defined as

$$\sigma_i^2 = \mathrm{Var}(X_i) = E[(X_i - \mu_i)^2] \tag{3.83}$$

The covariance of $X_i$ and $X_j$ is defined as

$$\sigma_{ij} = \text{Cov}(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] \tag{3.84}$$

The correlation coefficient of $X_i$ and $X_j$ is defined as

$$\rho_{ij} = \frac{\text{Cov}(X_i, X_j)}{\sigma_i \sigma_j} = \frac{\sigma_{ij}}{\sigma_i \sigma_j} \tag{3.85}$$

## 3.10 SPECIAL DISTRIBUTIONS

### A. Multinomial Distribution:

The multinomial distribution is an extension of the binomial distribution. An experiment is termed a multinomial trial with parameters $p_1, p_2, \ldots, p_k$, if it has the following conditions:

1. The experiment has $k$ possible outcomes that are mutually exclusive and exhaustive, say $A_1, A_2, \ldots, A_k$.

2. $P(A_i) = p_i \qquad i = 1, \ldots, k \qquad \text{and} \qquad \sum_{i=1}^{k} p_i = 1$ (3.86)

Consider an experiment which consists of $n$ repeated, independent, multinomial trials with parameters $p_1, p_2, \ldots, p_k$. Let $X_i$ be the r.v. denoting the number of trials which result in $A_i$. Then $(X_1, X_2, \ldots, X_k)$ is called the *multinomial* r.v. with parameters $(n, p_1, p_2, \ldots, p_k)$ and its pmf is given by (Prob. 3.46)

$$p_{X_1 X_2 \cdots X_k}(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k} \tag{3.87}$$

for $x_i = 0, 1, \ldots, n$, $i = 1, \ldots, k$, such that $\sum_{i=1}^{k} x_i = n$.

Note that when $k = 2$, the multinomial distribution reduces to the binomial distribution.

### B. Bivariate Normal Distribution:

A bivariate r.v. $(X, Y)$ is said to be a bivariate *normal* (or *gaussian*) r.v. if its joint pdf is given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_x \sigma_y (1 - \rho^2)^{1/2}} \exp[-\tfrac{1}{2} q(x, y)] \tag{3.88}$$

where

$$q(x, y) = \frac{1}{1 - \rho^2}\left[\left(\frac{x - \mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x - \mu_x}{\sigma_x}\right)\left(\frac{y - \mu_y}{\sigma_y}\right) + \left(\frac{y - \mu_y}{\sigma_y}\right)^2\right] \tag{3.89}$$

and $\mu_x$, $\mu_y$, $\sigma_x^2$, $\sigma_y^2$ are the means and variances of $X$ and $Y$, respectively. It can be shown that $\rho$ is the correlation coefficient of $X$ and $Y$ (Prob. 3.50) and that $X$ and $Y$ are independent when $\rho = 0$ (Prob. 3.49).

### C. *N*-variate Normal Distribution:

Let $(X_1, \ldots, X_n)$ be an $n$-variate r.v. defined on a sample space $S$. Let $\mathbf{X}$ be an $n$-dimensional *random vector* expressed as an $n \times 1$ matrix:

$$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \tag{3.90}$$

Let $\mathbf{x}$ be an $n$-dimensional vector ($n \times 1$ matrix) defined by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \tag{3.91}$$

The $n$-variate r.v. $(X_1, \ldots, X_n)$ is called an $n$-variate normal r.v. if its joint pdf is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} |\det K|^{1/2}} \exp\left[ -\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T K^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right] \tag{3.92}$$

where $T$ denotes the "transpose," $\boldsymbol{\mu}$ is the *vector mean*, $K$ is the *covariance matrix* given by

$$\boldsymbol{\mu} = E[\mathbf{X}] = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} \tag{3.93}$$

$$K = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} \qquad \sigma_{ij} = \mathrm{Cov}(X_i, X_j) \tag{3.94}$$

and $\det K$ is the determinant of the matrix $K$. Note that $f_{\mathbf{X}}(\mathbf{x})$ stands for $f_{X_1 \ldots X_n}(x_1, \ldots, x_n)$.

# Solved Problems

## BIVARIATE RANDOM VARIABLES AND JOINT DISTRIBUTION FUNCTIONS

**3.1.** Consider an experiment of tossing a fair coin twice. Let $(X, Y)$ be a bivariate r.v., where $X$ is the number of heads that occurs in the two tosses and $Y$ is the number of tails that occurs in the two tosses.

(*a*) What is the range $R_X$ of $X$?

(*b*) What is the range $R_Y$ of $Y$?

(*c*) Find and sketch the range $R_{XY}$ of $(X, Y)$.

(*d*) Find $P(X = 2, Y = 0)$, $P(X = 0, Y = 2)$, and $P(X = 1, Y = 1)$.

The sample space $S$ of the experiment is

$$S = \{HH, HT, TH, TT\}$$

(*a*) $R_X = \{0, 1, 2\}$

(*b*) $R_Y = \{0, 1, 2\}$

(*c*) $R_{XY} = \{(2, 0), (1, 1), (0, 2)\}$ which is sketched in Fig. 3-2.

(*d*) Since the coin is fair, we have

$$P(X = 2, Y = 0) = P\{HH\} = \tfrac{1}{4}$$
$$P(X = 0, Y = 2) = P\{TT\} = \tfrac{1}{4}$$
$$P(X = 1, Y = 1) = P\{HT, TH\} = \tfrac{1}{2}$$

**3.2.** Consider a bivariate r.v. $(X, Y)$. Find the region of the $xy$ plane corresponding to the events

$$A = \{X + Y \le 2\} \qquad B = \{X^2 + Y^2 < 4\}$$
$$C = \{\min(X, Y) \le 2\} \qquad D = \{\max(X, Y) \le 2\}$$

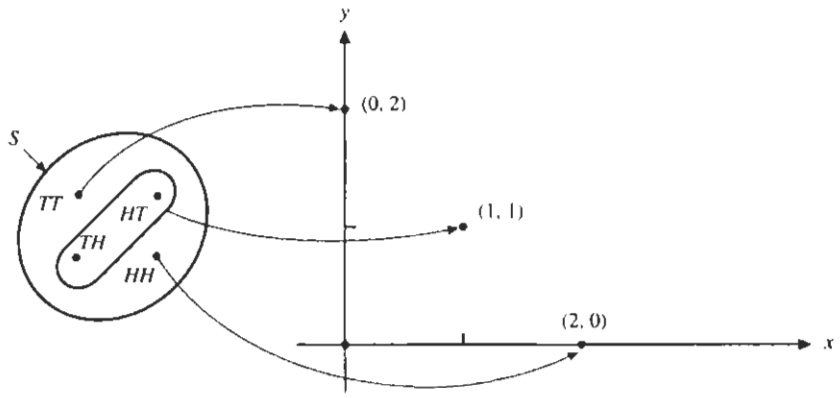



Fig. 3-2

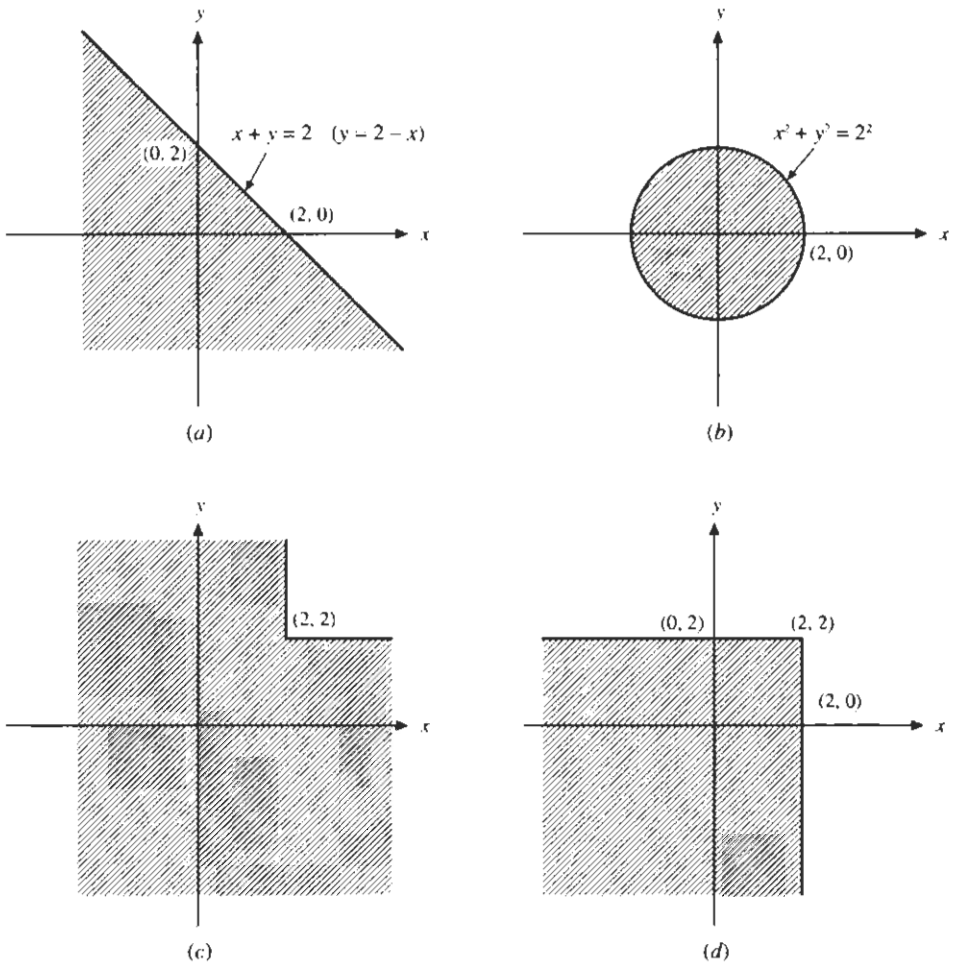



Fig. 3-3 Regions corresponding to certain joint events.

The region corresponding to event $A$ is expressed by $x + y \leq 2$, which is shown in Fig. 3-3($a$), that is, the region below and including the straight line $x + y = 2$.

The region corresponding to event $B$ is expressed by $x^2 + y^2 < 2^2$, which is shown in Fig. 3-3($b$), that is, the region within the circle with its center at the origin and radius 2.

The region corresponding to event $C$ is shown in Fig. 3-3($c$), which is found by noting that

$$\{\min(X, Y) \leq 2\} = (X \leq 2) \cup (Y \leq 2)$$

The region corresponding to event $D$ is shown in Fig. 3-3($d$), which is found by noting that

$$\{\max(X, Y) \leq 2\} = (X \leq 2) \cap (Y \leq 2)$$

**3.3.** Verify Eqs. (*3.7*), (*3.8a*), and (*3.8b*).

Since $\{X \leq \infty, Y \leq \infty\} = S$ and by Eq. (*1.22*),

$$P(X \leq \infty, Y \leq \infty) = F_{XY}(\infty, \infty) = P(S) = 1$$

Next, as we know, from Eq. (*2.8*),

$$P(X = -\infty) = P(Y = -\infty) = 0$$

Since $(X = -\infty, Y \leq y) \subset (X = -\infty)$ and $(X \leq x, Y \leq -\infty) \subset (Y = -\infty)$

and by Eq. (*1.27*), we have

$$P(X = -\infty, Y \leq y) = F_{XY}(-\infty, y) = 0$$
$$P(X \leq x, Y = -\infty) = F_{XY}(x, -\infty) = 0$$

**3.4.** Verify Eqs. (*3.10*) and (*3.11*).

Clearly $(X \leq x_2, Y \leq y) = (X \leq x_1, Y \leq y) \cup (x_1 < X \leq x_2, Y \leq y)$

The two events on the right-hand side are disjoint; hence by Eq. (*1.23*),

$$P(X \leq x_2, Y \leq y) = P(X \leq x_1, Y \leq y) + P(x_1 < X \leq x_2, Y \leq y)$$

or
$$P(x_1 < X \leq x_2, Y \leq y) = P(X \leq x_2, Y \leq y) - P(X \leq x_1, Y \leq y)$$
$$= F_{XY}(x_2, y) - F_{XY}(x_1, y)$$

Similarly,

$$(X \leq x, Y \leq y_2) = (X \leq x, Y \leq y_1) \cup (X \leq x, y_1 < Y \leq y_2)$$

Again the two events on the right-hand side are disjoint, hence

$$P(X \leq x, Y \leq y_2) = P(X \leq x, Y \leq y_1) + P(X \leq x, y_1 < Y \leq y_2)$$

or
$$P(X \leq x, y_1 < Y \leq y_2) = P(X \leq x, Y \leq y_2) - P(X \leq x, Y \leq y_1)$$
$$= F_{XY}(x, y_2) - F_{XY}(x, y_1)$$

**3.5.** Verify Eq. (*3.12*).

Clearly

$$(x_1 < X \leq x_2, Y \leq y_2) = (x_1 < X \leq x_2, Y \leq y_1) \cup (x_1 < X \leq x_2, y_1 < Y \leq y_2)$$

The two events on the right-hand side are disjoint; hence

$$P(x_1 < X \leq x_2, Y \leq y_2) = P(x_1 < X \leq x_2, Y \leq y_1) + P(x_1 < X \leq x_2, y_1 < Y \leq y_2)$$

Then using Eq. (*3.10*), we obtain

$$\begin{aligned} P(x_1 < X \leq x_2, y_1 < Y \leq y_2) &= P(x_1 < X \leq x_2, Y \leq y_2) - P(x_1 < X \leq x_2, Y \leq y_1) \\ &= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - [F_{XY}(x_2, y_1) - F_{XY}(x_1, y_1)] \\ &= F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1) \end{aligned} \tag{3.95}$$

Since the probability must be nonnegative, we conclude that

$$F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1) \geq 0$$

if $x_2 \geq x_1$ and $y_2 \geq y_1$.

**3.6.** Consider a function

$$F(x, y) = \begin{cases} 1 - e^{-(x+y)} & 0 \leq x < \infty, 0 \leq y < \infty \\ 0 & \text{otherwise} \end{cases}$$

Can this function be a joint cdf of a bivariate r.v. $(X, Y)$?

It is clear that $F(x, y)$ satisfies properties 1 to 5 of a cdf [Eqs. (3.5) to (3.9)]. But substituting $F(x, y)$ in Eq. (3.12) and setting $x_2 = y_2 = 2$ and $x_1 = y_1 = 1$, we get

$$\begin{aligned} F(2, 2) - F(1, 2) - F(2, 1) + F(1, 1) &= (1 - e^{-4}) - (1 - e^{-3}) - (1 - e^{-3}) + (1 - e^{-2}) \\ &= -e^{-4} + 2e^{-3} - e^{-2} = -(e^{-2} - e^{-1})^2 < 0 \end{aligned}$$

Thus, property 7 [Eq. (3.12)] is not satisfied. Hence $F(x, y)$ cannot be a joint cdf.

**3.7.** Consider a bivariate r.v. $(X, Y)$. Show that if $X$ and $Y$ are independent, then every event of the form $(a < X \leq b)$ is independent of every event of the form $(c < Y \leq d)$.

By definition (3.4), if $X$ and $Y$ are independent, we have

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

Setting $x_1 = a$, $x_2 = b$, $y_1 = c$, and $y_2 = d$ in Eq. (3.95) (Prob. 3.5), we obtain

$$\begin{aligned} P(a < X \leq b, c < Y \leq d) &= F_{XY}(b, d) - F_{XY}(a, d) - F_{XY}(b, c) + F_{XY}(a, c) \\ &= F_X(b)F_Y(d) - F_X(a)F_Y(d) - F_X(b)F_Y(c) + F_X(a)F_Y(c) \\ &= [F_X(b) - F_X(a)][F_Y(d) - F_Y(c)] \\ &= P(a < X \leq b)P(c < Y \leq d) \end{aligned}$$

which indicates that event $(a < X \leq b)$ and event $(c < Y \leq d)$ are independent [Eq. (1.46)].

**3.8.** The joint cdf of a bivariate r.v. $(X, Y)$ is given by

$$F_{XY}(x, y) = \begin{cases} (1 - e^{-\alpha x})(1 - e^{-\beta y}) & x \geq 0, y \geq 0, \alpha, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Find the marginal cdf's of $X$ and $Y$.

(*b*) Show that $X$ and $Y$ are independent.

(*c*) Find $P(X \leq 1, Y \leq 1)$, $P(X \leq 1)$, $P(Y > 1)$, and $P(X > x, Y > y)$.

(*a*) By Eqs. (3.13) and (3.14), the marginal cdf's of $X$ and $Y$ are

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 1 - e^{-\beta y} & y \geq 0 \\ 0 & y < 0 \end{cases}$$

(*b*) Since $F_{XY}(x, y) = F_X(x)F_Y(y)$, $X$ and $Y$ are independent.

(*c*) $P(X \leq 1, Y \leq 1) = F_{XY}(1, 1) = (1 - e^{-\alpha})(1 - e^{-\beta})$

$P(X \leq 1) = F_X(1) = (1 - e^{-\alpha})$

$P(Y > 1) = 1 - P(Y \leq 1) = 1 - F_Y(1) = e^{-\beta}$

By De Morgan's law (*1.15*), we have

$$\overline{(X > x) \cap (Y > y)} = \overline{(X > x)} \cup \overline{(Y > y)} = (X \leq x) \cup (Y \leq y)$$

Then by Eq. (*1.29*),

$$\begin{aligned} P[\overline{(X > x) \cap (Y > y)}] &= P(X \leq x) + P(Y \leq y) - P(X \leq x, Y \leq y) \\ &= F_X(x) + F_Y(y) - F_{XY}(x, y) \\ &= (1 - e^{-\alpha x}) + (1 - e^{-\beta y}) - (1 - e^{-\alpha x})(1 - e^{-\beta y}) \\ &= 1 - e^{-\alpha x} e^{-\beta y} \end{aligned}$$

Finally, by Eq. (*1.25*), we obtain

$$P(X > x, Y > y) = 1 - P[\overline{(X > x) \cap (Y > y)}] = e^{-\alpha x} e^{-\beta y}$$

**3.9.** The joint cdf of a bivariate r.v. $(X, Y)$ is given by

$$F_{XY}(x, y) = \begin{cases} 0 & x < 0 \quad \text{or} \quad y < 0 \\ p_1 & 0 \leq x < a, \quad 0 \leq y < b \\ p_2 & x \geq a, \quad 0 \leq y < b \\ p_3 & 0 \leq x < a, \quad y \geq b \\ 1 & x \geq a, \quad y \geq b \end{cases}$$

(*a*) Find the marginal cdf's of $X$ and $Y$.

(*b*) Find the conditions on $p_1$, $p_2$, and $p_3$ for which $X$ and $Y$ are independent.

(*a*) By Eq. (*3.13*), the marginal cdf of $X$ is given by

$$F_X(x) = F_{XY}(x, \infty) = \begin{cases} 0 & x < 0 \\ p_3 & 0 \leq x < a \\ 1 & x \geq a \end{cases}$$

By Eq. (*3.14*), the marginal cdf of $Y$ is given by

$$F_Y(y) = F_{XY}(\infty, y) = \begin{cases} 0 & y < 0 \\ p_2 & 0 \leq y < b \\ 1 & y \geq b \end{cases}$$

(*b*) For $X$ and $Y$ to be independent, by Eq. (*3.4*), we must have $F_{XY}(x, y) = F_X(x)F_Y(y)$. Thus, for $0 \leq x < a$, $0 \leq y < b$, we must have $p_1 = p_2 p_3$ for $X$ and $Y$ to be independent.

## DISCRETE BIVARIATE RANDOM VARIABLES—JOINT PROBABILITY MASS FUNCTIONS

**3.10.** Verify Eq. (*3.22*).

If $X$ and $Y$ are independent, then by Eq. (*1.46*),

$$p_{XY}(x_i, y_j) = P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) = p_X(x_i)p_Y(y_j)$$

**3.11.** Two fair dice are thrown. Consider a bivariate r.v. $(X, Y)$. Let $X = 0$ or 1 according to whether the first die shows an even number or an odd number of dots. Similarly, let $Y = 0$ or 1 according to the second die.

(*a*) Find the range $R_{XY}$ of $(X, Y)$.

(*b*) Find the joint pmf's of $(X, Y)$.

(*a*) The range of $(X, Y)$ is

$$R_{XY} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

(*b*) It is clear that $X$ and $Y$ are independent and

$$P(X = 0) = P(X = 1) = \tfrac{3}{6} = \tfrac{1}{2}$$
$$P(Y = 0) = P(Y = 1) = \tfrac{3}{6} = \tfrac{1}{2}$$

Thus $$p_{XY}(i, j) = P(X = i, Y = j) = P(X = i)P(Y = j) = \tfrac{1}{4} \qquad i, j = 0, 1$$

**3.12.** Consider the binary communication channel shown in Fig. 3-4 (Prob. 1.52). Let $(X, Y)$ be a bivariate r.v., where $X$ is the input to the channel and $Y$ is the output of the channel. Let $P(X = 0) = 0.5$, $P(Y = 1 \mid X = 0) = 0.1$, and $P(Y = 0 \mid X = 1) = 0.2$.

(*a*) Find the joint pmf's of $(X, Y)$.

(*b*) Find the marginal pmf's of $X$ and $Y$.

(*c*) Are $X$ and $Y$ independent?

(*a*) From the results of Prob. 1.52, we found that

$$P(X = 1) = 1 - P(X = 0) = 0.5$$
$$P(Y = 0 \mid X = 0) = 0.9 \qquad P(Y = 1 \mid X = 1) = 0.8$$

Then by Eq. (*1.41*), we obtain

$$P(X = 0, Y = 0) = P(Y = 0 \mid X = 0)P(X = 0) = 0.9(0.5) = 0.45$$
$$P(X = 0, Y = 1) = P(Y = 1 \mid X = 0)P(X = 0) = 0.1(0.5) = 0.05$$
$$P(X = 1, Y = 0) = P(Y = 0 \mid X = 1)P(X = 1) = 0.2(0.5) = 0.1$$
$$P(X = 1, Y = 1) = P(Y = 1 \mid X = 1)P(X = 1) = 0.8(0.5) = 0.4$$

Hence, the joint pmf's of $(X, Y)$ are

$$p_{XY}(0, 0) = 0.45 \qquad p_{XY}(0, 1) = 0.05$$
$$p_{XY}(1, 0) = 0.1 \qquad p_{XY}(1, 1) = 0.4$$

(*b*) By Eq. (*3.20*), the marginal pmf's of $X$ are

$$p_X(0) = \sum_{y_j} p_{XY}(0, y_j) = 0.45 + 0.05 = 0.5$$
$$p_X(1) = \sum_{y_j} p_{XY}(1, y_j) = 0.1 + 0.4 = 0.5$$

By Eq. (*3.21*), the marginal pmf's of $Y$ are

$$p_Y(0) = \sum_{x_i} p_{XY}(x_i, 0) = 0.45 + 0.1 = 0.55$$
$$p_Y(1) = \sum_{x_i} p_{XY}(x_i, 1) = 0.05 + 0.4 = 0.45$$

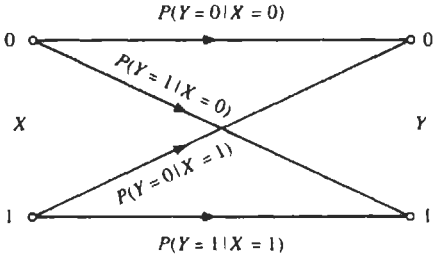



**Fig. 3-4** Binary communication channel.

(*c*) Now

$$p_X(0)p_Y(0) = 0.5(0.55) = 0.275 \neq p_{XY}(0, 0) = 0.45$$

Hence $X$ and $Y$ are not independent.

**3.13.** Consider an experiment of drawing randomly three balls from an urn containing two red, three white, and four blue balls. Let $(X, Y)$ be a bivariate r.v. where $X$ and $Y$ denote, respectively, the number of red and white balls chosen.

(*a*) Find the range of $(X, Y)$.

(*b*) Find the joint pmf's of $(X, Y)$.

(*c*) Find the marginal pmf's of $X$ and $Y$.

(*d*) Are $X$ and $Y$ independent?

(*a*) The range of $(X, Y)$ is given by

$$R_{XY} = \{(0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1)\}$$

(*b*) The joint pmf's of $(X, Y)$

$$p_{XY}(i, j) = P(X = i, Y = j) \qquad i = 0, 1, 2 \qquad j = 0, 1, 2, 3$$

are given as follows:

$$p_{XY}(0, 0) = \binom{4}{3}\Big/\binom{9}{3} = \frac{4}{84} \qquad p_{XY}(0, 1) = \binom{3}{1}\binom{4}{2}\Big/\binom{9}{3} = \frac{18}{84}$$

$$p_{XY}(0, 2) = \binom{3}{2}\binom{4}{1}\Big/\binom{9}{3} = \frac{12}{84} \qquad p_{XY}(0, 3) = \binom{3}{3}\Big/\binom{9}{3} = \frac{1}{84}$$

$$p_{XY}(1, 0) = \binom{2}{1}\binom{4}{2}\Big/\binom{9}{3} = \frac{12}{84} \qquad p_{XY}(1, 1) = \binom{2}{1}\binom{3}{1}\binom{4}{1}\Big/\binom{9}{3} = \frac{24}{84}$$

$$p_{XY}(1, 2) = \binom{2}{1}\binom{3}{2}\Big/\binom{9}{3} = \frac{6}{84}$$

$$p_{XY}(2, 0) = \binom{2}{2}\binom{4}{1}\Big/\binom{9}{3} = \frac{4}{84} \qquad p_{XY}(2, 1) = \binom{2}{2}\binom{3}{1}\Big/\binom{9}{3} = \frac{3}{84}$$

which are expressed in tabular form as in Table 3.1.

(*c*) The marginal pmf's of $X$ are obtained from Table 3.1 by computing the row sums, and the marginal pmf's of $Y$ are obtained by computing the column sums. Thus

$$p_X(0) = \tfrac{35}{84} \qquad p_X(1) = \tfrac{42}{84} \qquad p_X(2) = \tfrac{7}{84}$$

$$p_Y(0) = \tfrac{20}{84} \qquad p_Y(1) = \tfrac{45}{84} \qquad p_Y(2) = \tfrac{18}{84} \qquad p_Y(3) = \tfrac{1}{84}$$

**Table 3.1** $p_{XY}(i, j)$

| | $j$ | | | |
|---|---|---|---|---|
| $i$ | 0 | 1 | 2 | 3 |
| 0 | $\frac{4}{84}$ | $\frac{18}{84}$ | $\frac{12}{84}$ | $\frac{1}{84}$ |
| 1 | $\frac{12}{84}$ | $\frac{24}{84}$ | $\frac{6}{84}$ | 0 |
| 2 | $\frac{4}{84}$ | $\frac{3}{84}$ | 0 | 0 |

(*d*) Since

$$p_{XY}(0, 0) = \tfrac{4}{84} \neq p_X(0)p_Y(0) = \tfrac{35}{84}(\tfrac{20}{84})$$

$X$ and $Y$ are not independent.

**3.14.** The joint pmf of a bivariate r.v. $(X, Y)$ is given by

$$p_{XY}(x_i, y_j) = \begin{cases} k(2x_i + y_j) & x_i = 1, 2;\ y_j = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the marginal pmf's of $X$ and $Y$.

(*c*) Are $X$ and $Y$ independent?

(*a*) By Eq. (*3.17*),

$$\sum_{x_i}\sum_{y_j} p_{XY}(x_i, y_j) = \sum_{x_i=1}^{2}\sum_{y_j=1}^{2} k(2x_i + y_j)$$
$$= k[(2 + 1) + (2 + 2) + (4 + 1) + (4 + 2)] = k(18) = 1$$

Thus, $k = \frac{1}{18}$.

(*b*) By Eq. (*3.20*), the marginal pmf's of $X$ are

$$p_X(x_i) = \sum_{y_j} p_{XY}(x_i, y_j) = \sum_{y_j=1}^{2} \tfrac{1}{18}(2x_i + y_j)$$
$$= \tfrac{1}{18}(2x_i + 1) + \tfrac{1}{18}(2x_i + 2) = \tfrac{1}{18}(4x_i + 3) \qquad x_i = 1, 2$$

By Eq. (*3.21*), the marginal pmf's of $Y$ are

$$p_Y(y_j) = \sum_{x_i} p_{XY}(x_i, y_j) = \sum_{x_i=1}^{2} \tfrac{1}{18}(2x_i + y_j)$$
$$= \tfrac{1}{18}(2 + y_j) + \tfrac{1}{18}(4 + y_j) = \tfrac{1}{18}(2y_j + 6) \qquad y_j = 1, 2$$

(*c*) Now $p_X(x_i)p_Y(y_j) \neq p_{XY}(x_i, y_j)$; hence $X$ and $Y$ are not independent.

**3.15.** The joint pmf of a bivariate r.v. $(X, Y)$ is given by

$$p_{XY}(x_i, y_j) = \begin{cases} kx_i^2 y_j & x_i = 1, 2;\ y_j = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the marginal pmf's of $X$ and $Y$.

(*c*) Are $X$ and $Y$ independent?

(*a*) By Eq. (*3.17*),

$$\sum_{x_i}\sum_{y_j} p_{XY}(x_i, y_j) = \sum_{x_i=1}^{2}\sum_{y_j=1}^{3} kx_i^2 y_j$$
$$= k(1 + 2 + 3 + 4 + 8 + 12) = k(30) = 1$$

Thus, $k = \frac{1}{30}$.

(b) By Eq. (*3.20*), the marginal pmf's of $X$ are

$$p_X(x_i) = \sum_{y_j} p_{XY}(x_i, y_j) = \sum_{y_j=1}^{3} \tfrac{1}{30}x_i^2 y_j = \tfrac{1}{5}x_i^2 \qquad x_i = 1, 2$$

By Eq. (3.21), the marginal pmf's of $Y$ are

$$p_Y(y_j) = \sum_{x_i} p_{XY}(x_i, y_j) = \sum_{x_i=1}^{2} \tfrac{1}{30} x_i^2 y_j = \tfrac{1}{6} y_j \qquad y_j = 1, 2, 3$$

(c) Now

$$p_X(x_i) p_Y(y_j) = \tfrac{1}{30} x_i^2 y_j = p_{XY}(x_i, y_j)$$

Hence $X$ and $Y$ are independent.

**3.16.** Consider an experiment of tossing two coins three times. Coin $A$ is fair, but coin $B$ is not fair, with $P(H) = \frac{1}{4}$ and $P(T) = \frac{3}{4}$. Consider a bivariate r.v. $(X, Y)$, where $X$ denotes the number of heads resulting from coin $A$ and $Y$ denotes the number of heads resulting from coin $B$.

(a) Find the range of $(X, Y)$.

(b) Find the joint pmf's of $(X, Y)$.

(c) Find $P(X = Y)$, $P(X > Y)$, and $P(X + Y \leq 4)$.

(a) The range of $(X, Y)$ is given by

$$R_{XY} = \{(i, j): i, j = 0, 1, 2, 3\}$$

(b) It is clear that the r.v.'s $X$ and $Y$ are independent, and they are both binomial r.v.'s with parameters $(n, p) = (3, \frac{1}{2})$ and $(n, p) = (3, \frac{1}{4})$, respectively. Thus, by Eq. (2.36), we have

$$p_X(0) = P(X = 0) = \binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8} \qquad p_X(1) = P(X = 1) = \binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$p_X(2) = P(X = 2) = \binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8} \qquad p_X(3) = P(X = 3) = \binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$P_Y(0) = P(Y = 0) = \binom{3}{0}\left(\frac{1}{4}\right)^0\left(\frac{3}{4}\right)^3 = \frac{27}{64} \qquad p_Y(1) = P(Y = 1) = \binom{3}{1}\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2 = \frac{27}{64}$$

$$p_Y(2) = P(Y = 2) = \binom{3}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right) = \frac{9}{64} \qquad P_Y(3) = P(Y = 3) = \binom{3}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^0 = \frac{1}{64}$$

Since $X$ and $Y$ are independent, the joint pmf's of $(X, Y)$ are

$$p_{XY}(i, j) = p_X(i) p_Y(j) \qquad i, j = 0, 1, 2, 3$$

which are tabulated in Table 3.2.

(c) From Table 3.2, we have

$$P(X = Y) = \sum_{i=0}^{3} p_{XY}(i, i) = \tfrac{1}{512}(27 + 81 + 27 + 1) = \tfrac{136}{512}$$

$$P(X > Y) = \sum_{i>j}^{3} p_{XY}(i, j) = P_{XY}(1, 0) + P_{XY}(2, 0) + P_{XY}(3, 0) + p_{XY}(2, 1) + p_{XY}(3, 1) + p_{XY}(3, 2)$$
$$= \tfrac{1}{512}(81 + 81 + 27 + 81 + 27 + 9) = \tfrac{306}{512}$$

**Table 3.2 $p_{XY}(i, j)$**

| | $j$ | | | |
|---|---|---|---|---|
| $i$ | 0 | 1 | 2 | 3 |
| 0 | $\frac{27}{512}$ | $\frac{27}{512}$ | $\frac{9}{512}$ | $\frac{1}{512}$ |
| 1 | $\frac{81}{512}$ | $\frac{81}{512}$ | $\frac{27}{512}$ | $\frac{3}{512}$ |
| 2 | $\frac{81}{512}$ | $\frac{81}{512}$ | $\frac{27}{512}$ | $\frac{3}{512}$ |
| 3 | $\frac{27}{512}$ | $\frac{27}{512}$ | $\frac{9}{512}$ | $\frac{1}{512}$ |

$$P(X + Y > 4) = \sum_{i+j>4} p_{XY}(i, j) = P_{XY}(2, 3) + P_{XY}(3, 2) + P_{XY}(3, 3)$$
$$= \tfrac{1}{512}(3 + 9 + 1) = \tfrac{13}{512}$$

Thus, $$P(X + y \le 4) = 1 - P(X + y > 4) = 1 - \tfrac{13}{512} = \tfrac{499}{512}$$

## CONTINUOUS BIVARIATE RANDOM VARIABLES—PROBABILITY DENSITY FUNCTIONS

**3.17.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} k(x + y) & 0 < x < 2,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the marginal pdf's of $X$ and $Y$.

(*c*) Are $X$ and $Y$ independent?

(*a*) By Eq. (*3.26*),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k\int_0^2\int_0^2 (x + y)\,dx\,dy$$
$$= k\int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_{x=0}^{x=2} dy$$
$$= k\int_0^2 (2 + 2y)\,dy = 8k = 1$$

Thus $k = \frac{1}{8}$.

(*b*) By Eq. (*3.30*), the marginal pdf of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy = \frac{1}{8}\int_0^2 (x + y)\,dy$$
$$= \frac{1}{8}\left(xy + \frac{y^2}{2}\right)\Big|_{y=0}^{y=2} = \begin{cases} \frac{1}{4}(x + 1) & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Since $f_{XY}(x, y)$ is symmetric with respect to $x$ and $y$, the marginal pdf of $Y$ is

$$f_Y(y) = \begin{cases} \frac{1}{4}(y + 1) & 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

(*c*) Since $f_{XY}(x, y) \neq f_X(x)f_Y(y)$, $X$ and $Y$ are not independent.

**3.18.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} kxy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Are $X$ and $Y$ independent?

(*c*) Find $P(X + Y < 1)$.

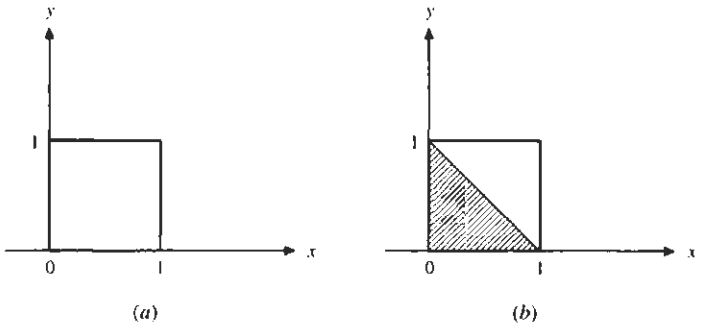



Fig. 3-5

(*a*) The range space $R_{XY}$ is shown in Fig. 3-5(*a*). By Eq. (*3.26*),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k\int_0^1\int_0^1 xy\,dx\,dy = k\int_0^1 y\left(\frac{x^2}{2}\Big|_0^1\right)dy$$
$$= k\int_0^1 \frac{y}{2}\,dy = \frac{k}{4} = 1$$

Thus $k = 4$.

(*b*) To determine whether $X$ and $Y$ are independent, we must find the marginal pdf's of $X$ and $Y$. By Eq. (*3.30*),

$$f_X(x) = \begin{cases}\displaystyle\int_0^1 4xy\,dy = 2x & 0 < x < 1\\ 0 & \text{otherwise}\end{cases}$$

By symmetry,

$$f_Y(y) = \begin{cases}2y & 0 < y < 1\\ 0 & \text{otherwise}\end{cases}$$

Since $f_{XY}(x, y) = f_X(x)f_Y(y)$, $X$ and $Y$ are independent.

(*c*) The region in the $xy$ plane corresponding to the event $(X + Y < 1)$ is shown in Fig. 3-5(*b*) as a shaded area. Then

$$P(X + Y < 1) = \int_0^1\int_0^{1-y} 4xy\,dx\,dy = \int_0^1 4y\left(\frac{x^2}{2}\Big|_0^{1-y}\right)dy$$
$$= \int_0^1 4y[\tfrac{1}{2}(1-y)^2]\,dy = \tfrac{1}{6}$$

**3.19.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases}kxy & 0 < x < y < 1\\ 0 & \text{otherwise}\end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Are $X$ and $Y$ independent?

(*a*) The range space $R_{XY}$ is shown in Fig. 3-6. By Eq. (*3.26*),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k\int_0^1\int_0^y xy\,dx\,dy = k\int_0^1 y\left(\frac{x^2}{2}\Big|_0^y\right)dy$$
$$= k\int_0^1 \frac{y^3}{2}\,dy = \frac{k}{8} = 1$$

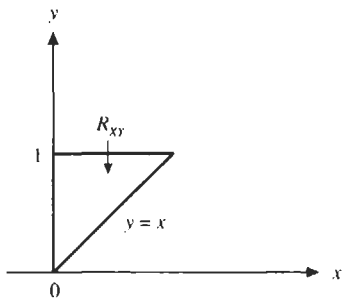



**Fig. 3-6**

Thus $k = 8$.

(*b*) By Eq. (*3.30*), the marginal pdf of $X$ is

$$f_X(x) = \begin{cases} \int_x^1 8xy\,dy = 4x(1 - x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

By Eq. (*3.31*), the marginal pdf of $Y$ is

$$f_Y(y) = \begin{cases} \int_0^y 8xy\,dx = 4y^3 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Since $f_{XY}(x, y) \neq f_X(x)f_Y(y)$, $X$ and $Y$ are not independent.

Note that if the range space $R_{XY}$ depends functionally on $x$ or $y$, then $X$ and $Y$ cannot be independent r.v.'s.

**3.20.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} k & 0 < y \leq x < 1 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Determine the value of $k$.

(*b*) Find the marginal pdf's of $X$ and $Y$.

(*c*) Find $P(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2})$.

(*a*) The range space $R_{XY}$ is shown in Fig. 3-7. By Eq. (*3.26*),

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k \iint_{R_{XY}} dx\,dy = k \times \text{area}(R_{XY}) = k(\tfrac{1}{2}) = 1$$

Thus $k = 2$.

(*b*) By Eq. (*3.30*), the marginal pdf of $X$ is

$$f_X(x) = \begin{cases} \int_0^x 2\,dy = 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

By Eq. (*3.31*), the marginal pdf of $Y$ is

$$f_Y(y) = \begin{cases} \int_y^1 2\,dx = 2(1 - y) & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

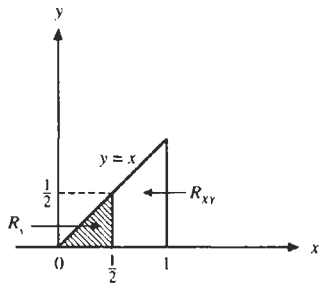



Fig. 3-7

(*c*) The region in the $xy$ plane corresponding to the event $(0 < X < \frac{1}{2}, 0 < Y < \frac{1}{2})$ is shown in Fig. 3-7 as the shaded area $R_s$. Then

$$P(0 < X < \tfrac{1}{2}, 0 < Y < \tfrac{1}{2}) = P(0 < X < \tfrac{1}{2}, 0 < Y < X)$$
$$= \iint_{R_s} f_{XY}(x, y)\, dx\, dy = 2 \iint_{R_s} dx\, dy = 2 \times \text{area}(R_s) = 2(\tfrac{1}{8}) = \tfrac{1}{4}$$

Note that the bivariate r.v. $(X, Y)$ is said to be uniformly distributed over the region $R_{XY}$ if its pdf is

$$f_{XY}(x, y) = \begin{cases} k & (x, y) \in R_{XY} \\ 0 & \text{otherwise} \end{cases} \tag{3.96}$$

where $k$ is a constant. Then by Eq. (*3.26*), the contant $k$ must be $k = 1/(\text{area of } R_{XY})$.

**3.21.** Suppose we select one point at random from within the circle with radius $R$. If we let the center of the circle denote the origin and define $X$ and $Y$ to be the coordinates of the point chosen (Fig. 3-8), then $(X, Y)$ is a uniform bivariate r.v. with joint pdf given by

$$f_{XY}(x, y) = \begin{cases} k & x^2 + y^2 \le R^2 \\ 0 & x^2 + y^2 > R^2 \end{cases}$$

where $k$ is a constant.

(*a*) Determine the value of $k$.

(*b*) Find the marginal pdf's of $X$ and $Y$.

(*c*) Find the probability that the distance from the origin of the point selected is not greater than $a$.

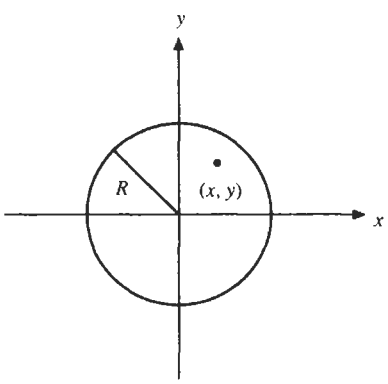



Fig. 3-8

(*a*) By Eq. (*3.26*),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k \iint_{x^2+y^2\le R^2} dx\,dy = k(\pi R^2) = 1$$

Thus $k = 1/\pi R^2$.

(*b*) By Eq. (*3.30*), the marginal pdf of $X$ is

$$f_X(x) = \frac{1}{\pi R^2}\int_{-\sqrt{(R^2-x^2)}}^{\sqrt{(R^2-x^2)}} dy = \frac{2}{\pi R^2}\sqrt{R^2-x^2} \qquad x^2 \le R^2$$

Hence

$$f_X(x) = \begin{cases} \dfrac{2}{\pi R^2}\sqrt{R^2-x^2} & |x| \le R \\ 0 & |x| > R \end{cases}$$

By symmetry, the marginal pdf of $Y$ is

$$f_Y(y) = \begin{cases} \dfrac{2}{\pi R^2}\sqrt{R^2-y^2} & |y| \le R \\ 0 & |y| > R \end{cases}$$

(*c*) For $0 \le a \le R$,

$$P(X^2+Y^2 \le a) = \iint_{x^2+y^2\le a^2} f_{XY}(x, y)\,dx\,dy$$

$$= \frac{1}{\pi R^2}\iint_{x^2+y^2\le a^2} dx\,dy = \frac{\pi a^2}{\pi R^2} = \frac{a^2}{R^2}$$

**3.22.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} ke^{-(ax+by)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a$ and $b$ are positive constants and $k$ is a constant.

(*a*) Determine the value of $k$.

(*b*) Are $X$ and $Y$ independent?

(*a*) By Eq. (*3.26*),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\,dy = k\int_0^{\infty}\int_0^{\infty} e^{-(ax+by)}\,dx\,dy$$

$$= k\int_0^{\infty} e^{-ax}\,dx\int_0^{\infty} e^{-by}\,dy = \frac{k}{ab} = 1$$

Thus $k = ab$.

(*b*) By Eq. (*3.30*), the marginal pdf of $X$ is

$$f_X(x) = abe^{-ax}\int_0^{\infty} e^{-by}\,dy = ae^{-ax} \qquad x > 0$$

By Eq. (*3.31*), the marginal pdf of $Y$ is

$$f_Y(y) = abe^{-by}\int_0^{\infty} e^{-ax}\,dx = be^{-bx} \qquad y > 0$$

Since $f_{XY}(x, y) = f_X(x)f_Y(y)$, $X$ and $Y$ are independent.

**3.23.** A manufacturer has been using two different manufacturing processes to make computer memory chips. Let $(X, Y)$ be a bivariate r.v., where $X$ denotes the time to failure of chips made by process $A$ and $Y$ denotes the time to failure of chips made by process $B$. Assuming that the joint pdf of $(X, Y)$ is

$$f_{XY}(x, y) = \begin{cases} abe^{-(ax+by)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a = 10^{-4}$ and $b = 1.2(10^{-4})$, determine $P(X > Y)$.

The region in the $xy$ plane corresponding to the event $(X > Y)$ is shown in Fig. 3-9 as the shaded area. Then

$$P(X > Y) = ab \int_0^\infty \int_0^x e^{-(ax+by)}\, dy\, dx$$

$$= ab \int_0^\infty e^{-ax} \left[ \int_0^x e^{-by}\, dy \right] dx = a \int_0^\infty e^{-ax}(1 - e^{-bx})\, dx$$

$$= \frac{b}{a+b} = \frac{1.2(10^{-4})}{(1 + 1.2)(10^{-4})} = 0.545$$

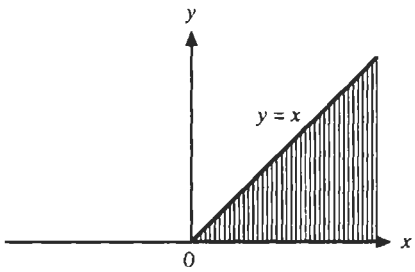



**Fig. 3-9**

**3.24.** A smooth-surface table is ruled with equidistant parallel lines a distance $D$ apart. A needle of length $L$, where $L \leq D$, is randomly dropped onto this table. What is the probability that the needle will intersect one of the lines? (This is known as *Buffon's* needle problem.)

We can determine the needle's position by specifying a bivariate r.v. $(X, \Theta)$, where $X$ is the distance from the middle point of the needle to the nearest parallel line and $\Theta$ is the angle from the vertical to the needle (Fig. 3-10). We interpret the statement "the needle is randomly dropped" to mean that both $X$ and $\Theta$ have uniform distributions and that $X$ and $\Theta$ are independent. The possible values of $X$ are between 0 and

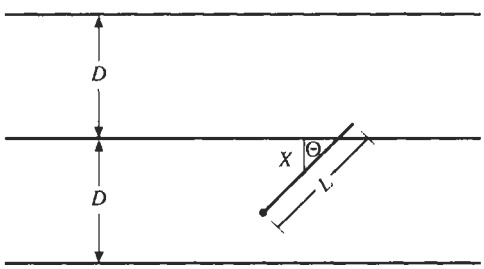



**Fig. 3-10** Buffon's needle problem.

$D/2$, and the possible values of $\Theta$ are between 0 and $\pi/2$. Thus, the joint pdf of $(X, \Theta)$ is

$$f_{X\Theta}(x, \theta) = f_X(x) f_\Theta(\theta) = \begin{cases} \dfrac{4}{\pi D} & 0 \le x \le \dfrac{D}{2},\ 0 \le \theta \le \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

From Fig. 3-10, we see that the condition for the needle to intersect a line is $X < L/2 \cos\theta$. Thus, the probability that the needle will intersect a line is

$$\begin{aligned} P\left(X < \frac{L}{2}\cos\theta\right) &= \int_0^{\pi/2} \int_0^{(L/2)\cos\theta} f_{X\Theta}(x, \theta)\, dx\, d\theta \\ &= \frac{4}{\pi D} \int_0^{\pi/2} \left[ \int_0^{(L/2)\cos\theta} dx \right] d\theta \\ &= \frac{4}{\pi D} \int_0^{\pi/2} \frac{L}{2}\cos\theta\, d\theta = \frac{2L}{\pi D} \end{aligned}$$

## CONDITIONAL DISTRIBUTIONS

**3.25.** Verify Eqs. (*3.36*) and (*3.41*).

(*a*) By Eqs. (*3.33*) and (*3.20*),

$$\sum_{y_j} p_{Y|X}(y_j \mid x_i) = \frac{\sum_{y_j} p_{YX}(x_i, y_j)}{p_X(x_i)} = \frac{p_X(x_i)}{p_X(x_i)} = 1$$

(*b*) Similarly, by Eqs. (*3.38*) and (*3.30*),

$$\int_{-\infty}^{\infty} f_{Y|X}(y \mid x) = \frac{\int_{-\infty}^{\infty} f_{YX}(x, y)\, dy}{f_X(x)} = \frac{f_X(x)}{f_X(x)} = 1$$

**3.26.** Consider the bivariate r.v. $(X, Y)$ of Prob. 3.14.

(*a*) Find the conditional pmf's $p_{Y|X}(y_j \mid x_i)$ and $p_{X|Y}(x_i \mid y_j)$.

(*b*) Find $P(Y = 2 \mid X = 2)$ and $P(X = 2 \mid Y = 2)$.

(*a*) From the results of Prob. 3.14, we have

$$p_{XY}(x_i, y_j) = \begin{cases} \frac{1}{18}(2x_i + y_j) & x_i = 1, 2;\ y_j = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(x_i) = \tfrac{1}{18}(4x_i + 3) \qquad x_i = 1, 2$$

$$p_Y(y_j) = \tfrac{1}{18}(2y_j + 6) \qquad y_j = 1, 2$$

Thus, by Eqs. (*3.33*) and (*3.34*),

$$p_{Y|X}(y_j \mid x_i) = \frac{2x_i + y_j}{4x_i + 3} \qquad y_j = 1, 2;\ x_i = 1, 2$$

$$p_{X|Y}(x_i \mid y_j) = \frac{2x_i + y_j}{2y_j + 6} \qquad x_i = 1, 2;\ y_j = 1, 2$$

(*b*) Using the results of part (*a*), we obtain

$$P(Y = 2 \mid X = 2) = p_{Y|X}(2 \mid 2) = \frac{2(2) + 2}{4(2) + 3} = \frac{6}{11}$$

$$P(X = 2 \mid Y = 2) = p_{X|Y}(2 \mid 2) = \frac{2(2) + 2}{2(2) + 6} = \frac{3}{5}$$

**3.27.** Find the conditional pmf's $p_{Y|X}(y_j \mid x_i)$ and $p_{X|Y}(x_i \mid y_j)$ for the bivariate r.v. $(X, Y)$ of Prob. 3.15.

From the results of Prob. 3.15, we have

$$p_{XY}(x_i, y_j) = \begin{cases} \frac{1}{30}x_i^2 y_j & x_i = 1, 2;\ y_j = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$p_X(x_i) = \tfrac{1}{5}x_i^2 \qquad x_i = 1, 2$$

$$p_Y(y_j) = \tfrac{1}{6}y_j \qquad y_j = 1, 2, 3$$

Thus, by Eqs. (*3.33*) and (*3.34*),

$$p_{Y|X}(y_j \mid x_i) = \frac{\frac{1}{30}x_i^2 y_j}{\frac{1}{5}x_i^2} = \tfrac{1}{6}y_j \qquad y_j = 1, 2, 3;\ x_i = 1, 2$$

$$p_{X|Y}(x_i \mid y_j) = \frac{\frac{1}{30}x_i^2 y_j}{\frac{1}{6}y_j} = \tfrac{1}{5}x_i^2 \qquad x_i = 1, 2;\ y_j = 1, 2, 3$$

Note that $P_{Y|X}(y_j \mid x_i) = p_Y(y_j)$ and $p_{X|Y}(x_i \mid y_j) = p_X(x_i)$, as must be the case since $X$ and $Y$ are independent, as shown in Prob. 3.15.

**3.28.** Consider the bivariate r.v. $(X, Y)$ of Prob. 3.17.

(*a*) Find the conditional pdf's $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$.

(*b*) Find $P(0 < Y < \frac{1}{2} \mid X = 1)$.

(*a*) From the results of Prob. 3.17, we have

$$f_{XY}(x, y) = \begin{cases} \frac{1}{8}(x + y) & 0 < x < 2,\ 0 < y < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \tfrac{1}{4}(x + 1) \qquad 0 < x < 2$$

$$f_Y(y) = \tfrac{1}{4}(y + 1) \qquad 0 < y < 2$$

Thus, by Eqs. (*3.38*) and (*3.39*),

$$f_{Y|X}(y \mid x) = \frac{\frac{1}{8}(x + y)}{\frac{1}{4}(x + 1)} = \frac{1}{2}\frac{x + y}{x + 1} \qquad 0 < x < 2,\ 0 < y < 2$$

$$f_{X|Y}(x \mid y) = \frac{\frac{1}{8}(x + y)}{\frac{1}{4}(y + 1)} = \frac{1}{2}\frac{x + y}{y + 1} \qquad 0 < x < 2,\ 0 < y < 2$$

(*b*) Using the results of part (*a*), we obtain

$$P(0 < Y < \tfrac{1}{2} \mid X = 1) = \int_0^{1/2} f_{Y|X}(y \mid x = 1)\,dy = \frac{1}{2}\int_0^{1/2} \left(\frac{1 + y}{2}\right) dy = \frac{5}{32}$$

**3.29.** Find the conditional pdf's $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$ for the bivariate r.v. $(X, Y)$ of Prob. 3.18.

From the results of Prob. 3.18, we have

$$f_{XY}(x, y) = \begin{cases} 4xy & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = 2x \qquad 0 < x < 1$$

$$f_Y(y) = 2y \qquad 0 < y < 1$$

Thus, by Eqs. (*3.38*) and (*3.39*),

$$f_{Y|X}(y \mid x) = \frac{4xy}{2x} = 2y \qquad 0 < y < 1,\ 0 < x < 1$$

$$f_{X|Y}(x \mid y) = \frac{4xy}{2y} = 2x \qquad 0 < x < 1,\ 0 < y < 1$$

Again note that $f_{Y|X}(y \mid x) = f_Y(y)$ and $f_{X|Y}(x \mid y) = f_X(x)$, as must be the case since $X$ and $Y$ are independent, as shown in Prob. 3.18.

**3.30.** Find the conditional pdf's $f_{Y|X}(y \mid x)$ and $f_{X|Y}(x \mid y)$ for the bivariate r.v. $(X, Y)$ of Prob. 3.20.

From the results of Prob. 3.20, we have

$$f_{XY}(x, y) = \begin{cases} 2 & 0 < y \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = 2x \qquad 0 < x < 1$$

$$f_Y(y) = 2(1 - y) \qquad 0 < y < 1$$

Thus, by Eqs. (*3.38*) and (*3.39*),

$$f_{Y|X}(y \mid x) = \frac{1}{x} \qquad y \le x < 1,\ 0 < x < 1$$

$$f_{X|Y}(x \mid y) = \frac{1}{1 - y} \qquad y \le x < 1,\ 0 < x < 1$$

**3.31.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} \dfrac{1}{y} e^{-x/y} e^{-y} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Show that $f_{XY}(x, y)$ satisfies Eq. (*3.26*).

(*b*) Find $P(X > 1 \mid Y = y)$.

(*a*) We have

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx\, dy &= \int_0^{\infty} \int_0^{\infty} \frac{1}{y} e^{-x/y} e^{-y}\, dx\, dy \\ &= \int_0^{\infty} \frac{1}{y} e^{-y} \left( \int_0^{\infty} e^{-x/y}\, dx \right) dy \\ &= \int_0^{\infty} \frac{1}{y} e^{-y} \left[ -y e^{-x/y} \Big|_{x=0}^{x=\infty} \right] dy \\ &= \int_0^{\infty} e^{-y}\, dy = 1 \end{aligned}$$

(*b*) First we must find the marginal pdf on $Y$. By Eq. (*3.31*),

$$\begin{aligned} f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx &= \frac{1}{y} e^{-y} \int_0^{\infty} e^{-x/y}\, dx \\ &= \frac{1}{y} e^{-y} \left[ -y e^{-x/y} \Big|_{x=0}^{x=\infty} \right] = e^{-y} \end{aligned}$$

By Eq. (*3.39*), the conditional pdf of $X$ is

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \begin{cases} \dfrac{1}{y} e^{-x/y} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} P(X > 1 \mid Y = y) = \int_1^{\infty} f_{X|Y}(x, y)\, dx &= \int_1^{\infty} \frac{1}{y} e^{-x/y}\, dx \\ &= -e^{-x/y} \Big|_{x=1}^{x=\infty} = e^{-1/y} \end{aligned}$$

## COVARIANCE AND CORRELATION COEFFICIENTS

**3.32.** Let $(X, Y)$ be a bivariate r.v. If $X$ and $Y$ are independent, show that $X$ and $Y$ are uncorrelated.

If $(X, Y)$ is a discrete bivariate r.v., then by Eqs. (*3.43*) and (*3.22*),

$$E(XY) = \sum_{y_j}\sum_{x_i} x_i y_j p_{XY}(x_i, y_j) = \sum_{y_j}\sum_{x_i} x_i y_j p_X(x_i) p_Y(y_j)$$

$$= \left[\sum_{x_i} x_i p_X(x_i)\right]\left[\sum_{y_j} y_j p_Y(y_j)\right] = E(X)E(Y)$$

If $(X, Y)$ is a continuous bivariate r.v., then by Eqs. (*3.43*) and (*3.32*),

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x, y)\, dx\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_X(x) f_Y(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx \int_{-\infty}^{\infty} y f_Y(y)\, dy = E(X)E(Y)$$

Thus, $X$ and $Y$ are uncorrelated by Eq. (*3.52*).

**3.33.** Suppose the joint pmf of a bivariate r.v. $(X, Y)$ is given by

$$p_{XY}(x_i, y_j) = \begin{cases} \frac{1}{3} & (0, 1), (1, 0), (2, 1) \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Are $X$ and $Y$ independent?

(*b*) Are $X$ and $Y$ uncorrelated?

(*a*) By Eq. (*3.20*), the marginal pmf's of $X$ are

$$p_X(0) = \sum_{y_j} p_{XY}(0, y_j) = p_{XY}(0, 1) = \tfrac{1}{3}$$

$$p_X(1) = \sum_{y_j} p_{XY}(1, y_j) = p_{XY}(1, 0) = \tfrac{1}{3}$$

$$p_X(2) = \sum_{y_j} p_{XY}(2, y_j) = p_{XY}(2, 1) = \tfrac{1}{3}$$

By Eq. (*3.21*), the marginal pmf's of $Y$ are

$$p_Y(0) = \sum_{x_i} p_{XY}(x_i, 0) = p_{XY}(1, 0) = \tfrac{1}{3}$$

$$p_Y(1) = \sum_{x_i} p_{XY}(x_i, 1) = p_{XY}(0, 1) + p_{XY}(2, 1) = \tfrac{2}{3}$$

and

$$p_{XY}(0, 1) = \tfrac{1}{3} \neq p_X(0)p_Y(1) = \tfrac{2}{9}$$

Thus $X$ and $Y$ are not independent.

(*b*) By Eq.s (*3.45a*), (*3.45b*), and (*3.43*), we have

$$E(X) = \sum_{x_i} x_i p_X(x_i) = (0)(\tfrac{1}{3}) + (1)(\tfrac{1}{3}) + (2)(\tfrac{1}{3}) = 1$$

$$E(Y) = \sum_{y_j} y_j p_y(y_j) = (0)(\tfrac{1}{3}) + (1)(\tfrac{2}{3}) = \tfrac{2}{3}$$

$$E(XY) = \sum_{y_j}\sum_{x_i} x_i y_j p_{XY}(x_i, y_j)$$

$$= (0)(1)(\tfrac{1}{3}) + (1)(0)(\tfrac{1}{3}) + (2)(1)(\tfrac{1}{3}) = \tfrac{2}{3}$$

Now by Eq. (*3.51*),

$$\mathrm{Cov}(X, Y) = E(XY) - E(X)E(Y) = \tfrac{2}{3} - (1)(\tfrac{2}{3}) = 0$$

Thus, $X$ and $Y$ are uncorrelated.

**3.34.** Let $(X, Y)$ be a bivariate r.v. with the joint pdf

$$f_{XY}(x, y) = \frac{x^2 + y^2}{4\pi} e^{-(x^2+y^2)/2} \qquad -\infty < x < \infty, \; -\infty < y < \infty$$

Show that $X$ and $Y$ are not independent but are uncorrelated.

By Eq. (*3.30*), the marginal pdf of $X$ is

$$\begin{aligned} f_X(x) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} (x^2 + y^2) e^{-(x^2+y^2)/2} \, dy \\ &= \frac{e^{-x^2/2}}{2\sqrt{2\pi}} \left( x^2 \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} y^2 e^{-y^2/2} \, dy \right) \end{aligned}$$

Noting that the integrand of the first integral in the above expression is the pdf of $N(0; 1)$ and the second integral in the above expression is the variance of $N(0; 1)$, we have

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} (x^2 + 1) e^{-x^2/2} \qquad -\infty < x < \infty$$

Since $f_{XY}(x, y)$ is symmetric in $x$ and $y$, we have

$$f_Y(y) = \frac{1}{2\sqrt{2\pi}} (y^2 + 1) e^{-y^2/2} \qquad -\infty < y < \infty$$

Now $f_{XY}(x, y) \neq f_X(x) f_Y(y)$, and hence $X$ and $Y$ are not independent. Next, by Eqs. (*3.47a*) and (*3.47b*),

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) \, dx = 0$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y) \, dy = 0$$

since for each integral the integrand is an odd function. By Eq. (*3.43*),

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) \, dx \, dy = 0$$

The integral vanishes because the contributions of the second and the fourth quadrants cancel those of the first and the third. Thus, $E(XY) = E(X)E(Y)$, and so $X$ and $Y$ are uncorrelated.

**3.35.** Let $(X, Y)$ be a bivariate r.v. Show that

$$[E(XY)]^2 \leq E(X^2)E(Y^2) \tag{3.97}$$

This is known as the *Cauchy-Schwarz inequality*.

Consider the expression $E[(X - \alpha Y)^2]$ for any two r.v.'s $X$ and $Y$ and a real variable $\alpha$. This expression, when viewed as a quadratic in $\alpha$, is greater than or equal to zero; that is,

$$E[(X - \alpha Y)^2] \geq 0$$

for any value of $\alpha$. Expanding this, we obtain

$$E(X^2) - 2\alpha E(XY) + \alpha^2 E(Y^2) \geq 0$$

Choose a value of $\alpha$ for which the left-hand side of this inequality is minimum,

$$\alpha = \frac{E(XY)}{E(Y^2)}$$

which results in the inequality

$$E(X^2) - \frac{[E(XY)]^2}{E(Y^2)} \geq 0 \qquad \text{or} \qquad [E(XY)]^2 \leq E(X^2)E(Y^2)$$

**3.36.** Verify Eq. (*3.54*).

From the Cauchy-Schwarz inequality [Eq. (*3.97*)], we have

$$\{E[(X - \mu_X)(Y - \mu_Y)]\}^2 \leq E[(X - \mu_X)^2]E[(Y - \mu_Y)^2]$$

or

$$\sigma_{XY}^2 \leq \sigma_X^2 \sigma_Y^2$$

Then

$$\rho_{XY}^2 = \frac{\sigma_{XY}^2}{\sigma_X^2 \sigma_Y^2} \leq 1$$

Since $\rho_{XY}$ is a real number, this implies

$$|\rho_{XY}| \leq 1 \qquad \text{or} \qquad -1 \leq \rho_{XY} \leq 1$$

**3.37.** Let $(X, Y)$ be the bivariate r.v. of Prob. 3.12.

(*a*) Find the mean and the variance of $X$.

(*b*) Find the mean and the variance of $Y$.

(*c*) Find the covariance of $X$ and $Y$.

(*d*) Find the correlation coefficient of $X$ and $Y$.

(*a*) From the results of Prob. 3.12, the mean and the variance of $X$ are evaluated as follows:

$$E(X) = \sum_{x_i} x_i p_X(x_i) = (0)(0.5) + (1)(0.5) = 0.5$$

$$E(X^2) = \sum_{x_i} x_i^2 p_X(x_i) = (0)^2(0.5) + (1)^2(0.5) = 0.5$$

$$\sigma_X^2 = E(X^2) - [E(X)]^2 = 0.5 - (0.5)^2 = 0.25$$

(*b*) Similarly, the mean and the variance of $Y$ are

$$E(Y) = \sum_{y_j} y_j p_Y(y_j) = (0)(0.55) + (1)(0.45) = 0.45$$

$$E(Y^2) = \sum_{y_j} y_j^2 p_Y(y_j) = (0)^2(0.55) + (1)^2(0.45) = 0.45$$

$$\sigma_Y^2 = E(Y^2) - [E(Y)]^2 = 0.45 - (0.45)^2 = 0.2475$$

(*c*) By Eq. (*3.43*),

$$\begin{aligned} E(XY) &= \sum_{y_j} \sum_{x_i} x_i y_j p_{XY}(x_i, y_j) \\ &= (0)(0)(0.45) + (0)(1)(0.05) + (1)(0)(0.1) + (1)(1)(0.4) \\ &= 0.4 \end{aligned}$$

By Eq. (*3.51*), the covariance of $X$ and $Y$ is

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0.4 - (0.5)(0.45) = 0.175$$

(*d*) By Eq. (*3.53*), the correlation coefficient of $X$ and $Y$ is

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{0.175}{\sqrt{(0.25)(0.2475)}} = 0.704$$

**3.38.** Suppose that a bivariate r.v. $(X, Y)$ is uniformly distributed over a unit circle (Prob. 3.21).

(*a*) Are $X$ and $Y$ independent?

(*b*) Are $X$ and $Y$ correlated?

(*a*) Setting $R = 1$ in the results of Prob. 3.21, we obtain

$$f_{XY}(x, y) = \begin{cases} \dfrac{1}{\pi} & x^2 + y^2 < 1 \\ 0 & x^2 + y^2 > 1 \end{cases}$$

$$f_X(x) = \frac{2}{\pi}\sqrt{1 - x^2} \qquad |x| < 1$$

$$f_Y(y) = \frac{2}{\pi}\sqrt{1 - y^2} \qquad |y| < 1$$

Since $f_{XY}(x, y) \neq f_X(x) f_Y(y)$, $X$ and $Y$ are not independent.

(*b*) By Eqs. (*3.47a*) and (*3.47b*), the means of $X$ and $Y$ are

$$E(X) = \frac{2}{\pi}\int_{-1}^{1} x\sqrt{1 - x^2}\, dx = 0$$

$$E(Y) = \frac{2}{\pi}\int_{-1}^{1} y\sqrt{1 - y^2}\, dy = 0$$

since each integrand is an odd function.

Next, by Eq. (*3.43*),

$$E(XY) = \frac{1}{\pi}\iint_{x^2 + y^2 < 1} xy\, dx\, dy = 0$$

The integral vanishes because the contributions of the second and the fourth quadrants cancel those of the first and the third. Hence, $E(XY) = E(X)E(Y) = 0$ and $X$ and $Y$ are uncorrelated.

## CONDITIONAL MEANS AND CONDITIONAL VARIANCES

**3.39.** Consider the bivariate r.v. $(X, Y)$ of Prob. 3.14 (or Prob. 3.26). Compute the conditional mean and the conditional variance of $Y$ given $x_i = 2$.

From Prob. 3.26, the conditional pmf $p_{Y|X}(y_j | x_i)$ is

$$p_{Y|X}(y_j | x_i) = \frac{2x_i + y_j}{4x_i + 3} \qquad y_j = 1, 2;\ x_i = 1, 2$$

Thus,

$$p_{Y|X}(y_j | 2) = \frac{4 + y_j}{11} \qquad y_j = 1, 2$$

and by Eqs. (*3.55*) and (*3.56*), the conditional mean and the conditional variance of $Y$ given $x_i = 2$ are

$$\mu_{Y|2} = E(Y | x_i = 2) = \sum_{y_j} y_j p_{Y|X}(y_j | 2) = \sum_{y_j} y_j\left(\frac{4 + y_j}{11}\right)$$

$$= (1)\left(\frac{5}{11}\right) + (2)\left(\frac{6}{11}\right) = \frac{17}{11} \approx 1.545$$

$$\sigma^2_{Y|2} = E\left[\left(Y - \frac{17}{11}\right)^2 | x_i = 2\right] = \sum_{y_j}\left(y_j - \frac{17}{11}\right)^2\left(\frac{4 + y_j}{11}\right)$$

$$= \left(\frac{-6}{11}\right)^2\left(\frac{5}{11}\right) + \left(\frac{5}{11}\right)^2\left(\frac{6}{11}\right) = \frac{330}{1331} \approx 0.248$$

**3.40.** Let $(X, Y)$ be the bivariate r.v. of Prob. 3.20 (or Prob. 3.30). Compute the conditional means $E(Y \mid x)$ and $E(X \mid y)$.

From Prob. 3.30,

$$f_{Y|X}(y \mid x) = \frac{1}{x} \qquad y \le x < 1,\ 0 < x < 1$$

$$f_{X|Y}(x \mid y) = \frac{1}{1-y} \qquad y \le x < 1,\ 0 < x < 1$$

By Eq. (*3.58*), the conditional mean of $Y$, given $X = x$, is

$$E(Y \mid x) = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\, dy = \int_0^x y\left(\frac{1}{x}\right) dy = \frac{y^2}{2x}\bigg|_{y=0}^{y=x} = \frac{x}{2} \qquad 0 < x < 1$$

Similarly, the conditional mean of $X$, given $Y = y$, is

$$E(X \mid y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx = \int_y^1 x\left(\frac{1}{1-y}\right) dx = \frac{x^2}{2(1-y)}\bigg|_{x=y}^{x=1} = \frac{1+y}{2} \qquad 0 < y < 1$$

Note that $E(Y \mid x)$ is a function of $x$ only and $E(X \mid y)$ is a function of $y$ only.

**3.41.** Let $(X, Y)$ be the bivariate r.v. of Prob. 3.20 (or Prob. 3.30). Compute the conditional variances $\mathrm{Var}(Y \mid x)$ and $\mathrm{Var}(X \mid y)$.

Using the results of Prob. 3.40 and Eq. (*3.59*), the conditional variance of $Y$, given $X = x$, is

$$\mathrm{Var}(Y \mid x) = E\{[Y - E(Y \mid x)]^2 \mid x\} = \int_{-\infty}^{\infty} \left(y - \frac{x}{2}\right)^2 f_{Y|X}(y \mid x)\, dy$$

$$= \int_0^x \left(y - \frac{x}{2}\right)^2 \left(\frac{1}{x}\right) dy = \frac{1}{3x}\left(y - \frac{x}{2}\right)^3 \bigg|_{y=0}^{y=x} = \frac{x^2}{12}$$

Similarly, the conditional variance of $X$, given $Y = y$, is

$$\mathrm{Var}(X \mid y) = E\{[X - E(X \mid y)]^2 \mid y\} = \int_{-\infty}^{\infty} \left(x - \frac{1+y}{2}\right)^2 f_{X|Y}(x \mid y)\, dx$$

$$= \int_y^1 \left(x - \frac{1+y}{2}\right)^2 \left(\frac{1}{1-y}\right) dx = \frac{1}{3(1-y)}\left(x - \frac{1+y}{2}\right)^3 \bigg|_{x=y}^{x=1} = \frac{(1-y)^2}{12}$$

## *N*-DIMENSIONAL RANDOM VECTORS

**3.42.** Let $(X_1, X_2, X_3, X_4)$ be a four-dimensional random vector, where $X_k$ $(k = 1, 2, 3, 4)$ are independent Poisson r.v.'s with parameter 2.

(*a*) Find $P(X_1 = 1, X_2 = 3, X_3 = 2, X_4 = 1)$.

(*b*) Find the probability that exactly one of the $X_k$'s equals zero.

(*a*) By Eq. (*2.40*), the pmf of $X_k$ is

$$p_{X_k}(i) = P(X_k = i) = e^{-2}\frac{2^i}{i!} \qquad i = 0, 1, \ldots \tag{3.98}$$

Since the $X_k$'s are independent, by Eq. (*3.80*),

$$P(X_1 = 1, X_2 = 3, X_3 = 2, X_4 = 1) = p_{X_1}(1)p_{X_2}(3)p_{X_3}(2)p_{X_4}(1)$$

$$= \left(\frac{e^{-2}2}{1!}\right)\left(\frac{e^{-2}2^3}{3!}\right)\left(\frac{e^{-2}2^2}{2!}\right)\left(\frac{e^{-2}2}{1!}\right) = \frac{e^{-8}2^7}{12} \approx 3.58(10^{-3})$$

(*b*) First, we find the probability that $X_k = 0$, $k = 1, 2, 3, 4$. From Eq. (*3.98*),

$$P(X_k = 0) = e^{-2} \qquad k = 1, 2, 3, 4$$

Next, we treat zero as "success." If $Y$ denotes the number of successes, then $Y$ is a binomial r.v. with parameters $(n, p) = (4, e^{-2})$. Thus, the probability that exactly one of the $X_k$'s equals zero is given by [Eq. (2.36)]

$$P(Y = 1) = \binom{4}{1}(e^{-2})(1 - e^{-2})^3 \approx 0.35$$

**3.43.** Let $(X, Y, Z)$ be a trivariate r.v., where $X$, $Y$, and $Z$ are independent uniform r.v.'s over (0, 1). Compute $P(Z \geq XY)$.

Since $X, Y, Z$ are independent and uniformly distributed over (0, 1), we have

$$f_{XYZ}(x, y, z) = f_X(x) f_Y(y) f_Z(z) = 1 \qquad 0 < x < 1, 0 < y < 1, 0 < z < 1$$

Then

$$P(Z \geq XY) = \iiint_{z > xy} f_{XYZ}(x, y, z)\, dx\, dy\, dz = \int_0^1 \int_0^1 \int_{xy}^1 dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 (1 - xy)\, dy\, dx = \int_0^1 \left(1 - \frac{x}{2}\right) dx = \frac{3}{4}$$

**3.44.** Let $(X, Y, Z)$ be a trivariate r.v. with joint pdf

$$f_{XYZ}(x, y, z) = \begin{cases} ke^{-(ax + by + cz)} & x > 0, y > 0, z > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $a, b, c > 0$ and $k$ are constants.

(*a*) Determine the value of $k$.

(*b*) Find the marginal joint pdf of $X$ and $Y$.

(*c*) Find the marginal pdf of $X$.

(*d*) Are $X$, $Y$, and $Z$ independent?

(*a*) By Eq. (3.76),

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dx\, dy\, dz = k \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-(ax + by + cz)}\, dx\, dy\, dz$$

$$= k \int_0^{\infty} e^{-ax}\, dx \int_0^{\infty} e^{-by}\, dy \int_0^{\infty} e^{-cz}\, dz = \frac{k}{abc} = 1$$

Thus $k = abc$.

(*b*) By Eq. (3.77), the marginal joint pdf of $X$ and $Y$ is

$$f_{XY}(x, y) = \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dz = abc \int_0^{\infty} e^{-(ax + by + cz)}\, dz$$

$$= abce^{-(ax + by)} \int_0^{\infty} e^{-cz}\, dz = abe^{-(ax + by)} \qquad x > 0, y > 0$$

(*c*) By Eq. (3.78), the marginal pdf of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dy\, dz = abc \int_0^{\infty} \int_0^{\infty} e^{-(ax + by + cz)}\, dy\, dz$$

$$= abce^{-ax} \int_0^{\infty} e^{-by}\, dy \int_0^{\infty} e^{-cz}\, dz = ae^{-ax} \qquad x > 0$$

(*d*) Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dx\, dz = be^{-by} \qquad y > 0$$

$$f_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XYZ}(x, y, z)\, dx\, dy = ce^{-cz} \qquad z > 0$$

Since $f_{XYZ}(x, y, z) = f_X(x) f_Y(y) f_z(z)$, $X$, $Y$, and $Z$ are independent.

**3.45.** Show that

$$f_{XYZ}(x, y, z) = f_{Z|X,Y}(z \mid x, y) f_{Y|X}(y \mid x) f_X(x) \tag{3.99}$$

By definition (*3.79*),

$$f_{Z|X,Y}(z \mid x, y) = \frac{f_{XYZ}(x, y, z)}{f_{XY}(x, y)}$$

Hence

$$f_{XYZ}(x, y, z) = f_{Z|X,Y}(z \mid x, y) f_{XY}(x, y) \tag{3.100}$$

Now, by Eq. (*3.38*),

$$f_{XY}(x, y) = f_{Y|X}(y \mid x) f_X(x)$$

Substituting this expression into Eq. (*3.100*), we obtain

$$f_{XYZ}(x, y, z) = f_{Z|X,Y}(z \mid x, y) f_{Y|X}(y \mid x) f_X(x)$$

## SPECIAL DISTRIBUTIONS

**3.46.** Derive Eq. (*3.87*).

Consider a sequence of $n$ independent multinomial trials. Let $A_i$ ($i = 1, 2, \ldots, k$) be the outcome of a single trial. The r.v. $X_i$ is equal to the number of times $A_i$ occurs in the $n$ trials. If $x_1, x_2, \ldots, x_k$ are nonnegative integers such that their sum equals $n$, then for such a sequence the probability that $A_i$ occurs $x_i$ times, $i = 1, 2, \ldots, k$—that is, $P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k)$—can be obtained by counting the number of sequences containing exactly $x_1$ $A_1$'s, $x_2$ $A_2$'s, ..., $x_k$ $A_k$'s and multiplying by $p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$. The total number of such sequences is given by the number of ways we could lay out in a row $n$ things, of which $x_1$ are of one kind, $x_2$ are of a second kind, ..., $x_k$ are of a $k$th kind. The number of ways we could choose $x_1$ positions for the $A_1$'s is $\binom{n}{x_1}$; after having put the $A_1$'s in their position, the number of ways we could choose positions for the $A_2$'s is $\binom{n - x_1}{x_2}$, and so on. Thus, the total number of sequences with $x_1$ $A_1$'s, $x_2$ $A_2$'s, ..., $x_k$ $A_k$'s is given by

$$\begin{aligned}
&\binom{n}{x_1}\binom{n - x_1}{x_2}\binom{n - x_1 - x_2}{x_3} \cdots \binom{n - x_1 - x_2 - \cdots - x_{k-1}}{x_k} \\
&\quad = \frac{n!}{x_1!(n - x_1)!} \frac{(n - x_1)!}{x_2!(n - x_1 - x_2)!} \cdots \frac{(n - x_1 - x_2 - \cdots - x_{k-1})!}{x_k!\,0!} \\
&\quad = \frac{n!}{x_1!\, x_2! \cdots x_k!}
\end{aligned}$$

Thus, we obtain

$$p_{X_1 X_2 \cdots X_k}(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!\, x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

**3.47.** Suppose that a fair die is rolled seven times. Find the probability that 1 and 2 dots appear twice each; 3, 4, and 5 dots once each; and 6 dots not at all.

Let $(X_1, X_2, \ldots, X_6)$ be a six-dimensional random vector, where $X_i$ denotes the number of times $i$ dots appear in seven rolls of a fair die. Then $(X_1, X_2, \ldots, X_6)$ is a multinomial r.v. with parameters $(7, p_1, p_2, \ldots,$

$p_6$) where $p_i = \frac{1}{6}$ $(i = 1, 2, \ldots, 6)$. Hence, by Eq. (3.87),

$$P(X_1 = 2, X_2 = 2, X_3 = 1, X_4 = 1, X_5 = 1, X_6 = 0) = \frac{7!}{2!\,2!\,1!\,1!\,1!\,0!}\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^2\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^1\left(\frac{1}{6}\right)^0$$

$$= \frac{7!}{2!\,2!}\left(\frac{1}{6}\right)^7 = \frac{35}{6^5} \approx 0.0045$$

**3.48.** Show that the pmf of a multinomial r.v. given by Eq. (3.87) satisfies the condition (3.68); that is,

$$\sum\sum\cdots\sum p_{X_1X_2\cdots X_k}(x_1, x_2, \ldots, x_k) = 1 \tag{3.101}$$

where the summation is over the set of all nonnegative integers $x_1, x_2, \ldots, x_k$ whose sum is $n$.

The *multinomial theorem* (which is an extension of the binomial theorem) states that

$$(a_1 + a_2 + \cdots + a_k)^n = \sum \binom{n}{x_1 x_2 \cdots x_k} a_1^{x_1} a_2^{x_2} \cdots a_k^{x_k} \tag{3.102}$$

where $x_1 + x_2 + \cdots + x_k = n$ and

$$\binom{n}{x_1 x_2 \cdots x_k} = \frac{n!}{x_1!\,x_2! \cdots x_k!}$$

is called the *multinomial coefficient*, and the summation is over the set of all nonnegative integers $x_1, x_2, \ldots, x_k$ whose sum is $n$.

Thus, setting $a_i = p_i$ in Eq. (3.102), we obtain

$$\sum\sum\cdots\sum p_{X_1X_2\cdots X_k}(x_1, x_2, \ldots, x_k) = (p_1 + p_2 + \cdots + p_k)^n = (1)^n = 1$$

**3.49.** Let $(X, Y)$ be a bivariate normal r.v. with its pdf given by Eq. (3.88).

(*a*) Find the marginal pdf's of $X$ and $Y$.

(*b*) Show that $X$ and $Y$ are independent when $\rho = 0$.

(*a*) By Eq. (3.30), the marginal pdf of $X$ is

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy$$

From Eqs. (3.88) and (3.89), we have

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y(1-\rho^2)^{1/2}} \exp\left[-\frac{1}{2}q(x, y)\right]$$

$$q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

Rewriting $q(x, y)$,

$$q(x, y) = \frac{1}{1-\rho^2}\left[\left(\frac{y-\mu_Y}{\sigma_Y}\right) - \rho\left(\frac{x-\mu_X}{\sigma_X}\right)\right]^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2$$

$$= \frac{1}{(1-\rho^2)\sigma_Y^2}\left[y - \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right]^2 + \left(\frac{x-\mu_X}{\sigma_X}\right)^2$$

Then $$f_X(x) = \frac{\exp\left[-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2\right]}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma_Y(1-\rho^2)^{1/2}}\exp\left[-\frac{1}{2}q_1(x, y)\right]dy$$

where $$q_1(x, y) = \frac{1}{(1-\rho^2)\sigma_Y^2}\left[y - \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)\right]^2$$

Comparing the integrand with Eq. (2.52), we see that the integrand is a normal pdf with mean $\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X)$ and variance $(1 - \rho^2)\sigma_Y{}^2$. Thus, the integral must be unity and we obtain

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left[\frac{-(x - \mu_X)^2}{2\sigma_X{}^2}\right] \tag{3.103}$$

In a similar manner, the marginal pdf of $Y$ is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left[\frac{-(y - \mu_Y)^2}{2\sigma_Y{}^2}\right] \tag{3.104}$$

(*b*) When $\rho = 0$, Eq. (3.88) reduces to

$$\begin{aligned} f_{XY}(x, y) &= \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{-\frac{1}{2}\left[\left(\frac{x - \mu_X}{\sigma_X}\right)^2 + \left(\frac{y - \mu_Y}{\sigma_Y}\right)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}\,\sigma_X} \exp\left[-\frac{1}{2}\left(\frac{x - \mu_X}{\sigma_X}\right)^2\right] \frac{1}{\sqrt{2\pi}\,\sigma_Y} \exp\left[-\frac{1}{2}\left(\frac{y - \mu_Y}{\sigma_Y}\right)^2\right] \\ &= f_X(x) f_Y(y) \end{aligned}$$

Hence, $X$ and $Y$ are independent.

**3.50.** Show that $\rho$ in Eq. (3.88) is the correlation coefficient of $X$ and $Y$.

By Eqs. (3.50) and (3.53), the correlation coefficient of $X$ and $Y$ is

$$\begin{aligned} \rho_{XY} &= E\left[\left(\frac{X - \mu_X}{\sigma_X}\right)\left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left(\frac{x - \mu_X}{\sigma_X}\right)\left(\frac{y - \mu_Y}{\sigma_Y}\right)\right] f_{XY}(x, y)\,dx\,dy \end{aligned} \tag{3.105}$$

where $f_{XY}(x, y)$ is given by Eq. (3.88). By making a change in variables $v = (x - \mu_X)/\sigma_X$ and $w = (y - \mu_Y)/\sigma_Y$, we can write Eq. (3.105) as

$$\begin{aligned} \rho_{XY} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} vw \frac{1}{2\pi(1 - \rho^2)^{1/2}} \exp\left[-\frac{1}{2(1 - \rho^2)}(v^2 - 2\rho vw + w^2)\right] dv\,dw \\ &= \int_{-\infty}^{\infty} \frac{w}{\sqrt{2\pi}} \left\{\int_{-\infty}^{\infty} \frac{v}{\sqrt{2\pi}(1 - \rho^2)^{1/2}} \exp\left[-\frac{(v - \rho w)^2}{2(1 - \rho^2)}\right] dv\right\} e^{-w^2/2}\,dw \end{aligned}$$

The term in the curly braces is identified as the mean of $V = N(\rho w; 1 - \rho^2)$, and so

$$\rho_{XY} = \int_{-\infty}^{\infty} \frac{w}{\sqrt{2\pi}} (\rho w) e^{-w^2/2}\,dw = \rho \int_{-\infty}^{\infty} w^2 \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\,dw$$

The last integral is the variance of $W = N(0; 1)$, and so it is equal to 1 and we obtain $\rho_{XY} = \rho$.

**3.51.** Let $(X, Y)$ be a bivariate normal r.v. with its pdf given by Eq. (3.88). Determine $E(Y \mid x)$.

By Eq. (3.58),

$$E(Y \mid x) = \int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\,dy \tag{3.106}$$

where

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} \tag{3.107}$$

Substituting Eqs. (3.88) and (3.103) into Eq. (3.107), and after some cancellation and rearranging, we obtain

$$f_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi}\,\sigma_Y(1 - \rho^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma_Y{}^2(1 - \rho^2)}\left[y - \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X) - \mu_Y\right]^2\right\}$$

which is equal to the pdf of a normal r.v. with mean $\mu_Y + \rho(\sigma_Y/\sigma_X)(x - \mu_X)$ and variance $(1 - \rho^2)\sigma_Y^2$. Thus, we get

$$E(Y \mid x) = \mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X) \tag{3.108}$$

Note that when $X$ and $Y$ are independent, then $\rho = 0$ and $E(Y \mid x) = \mu_Y = E(Y)$.

**3.52.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \frac{1}{2\sqrt{3}\pi} \exp\left[ -\tfrac{1}{3}(x^2 - xy + y^2 + x - 2y + 1) \right] \qquad -\infty < x, y < \infty$$

(*a*) Find the means of $X$ and $Y$.

(*b*) Find the variances of $X$ and $Y$.

(*c*) Find the correlation coefficient of $X$ and $Y$.

We note that the term in the bracket of the exponential is a quadratic function of $x$ and $y$, and hence $f_{XY}(x, y)$ could be a pdf of a bivariate normal r.v. If so, then it is simpler to solve equations for the various parameters. Now, the given joint pdf of $(X, Y)$ can be expressed as

$$f_{XY}(x, y) = \frac{1}{2\sqrt{3}\pi} \exp\left[ -\tfrac{1}{2}q(x, y) \right]$$

where

$$\begin{aligned} q(x, y) &= \tfrac{2}{3}(x^2 - xy + y^2 + x - 2y + 1) \\ &= \tfrac{2}{3}[x^2 - x(y - 1) + (y - 1)^2] \end{aligned}$$

Comparing the above expressions with Eqs. (*3.88*) and (*3.89*), we see that $f_{XY}(x, y)$ is the pdf of a bivariate normal r.v. with $\mu_X = 0$, $\mu_Y = 1$, and the following equations:

$$2\pi\sigma_X\sigma_Y\sqrt{1 - \rho^2} = 2\sqrt{3}\pi$$

$$(1 - \rho^2)\sigma_X^2 = (1 - \rho^2)\sigma_Y^2 = \tfrac{3}{2}$$

$$\frac{2\rho}{\sigma_X\sigma_Y(1 - \rho^2)} = \frac{2}{3}$$

Solving for $\sigma_X^2$, $\sigma_Y^2$, and $\rho$, we get

$$\sigma_X^2 = \sigma_Y^2 = 2 \qquad \text{and} \qquad \rho = \tfrac{1}{2}$$

Hence

(*a*) The mean of $X$ is zero, and the mean of $Y$ is 1.

(*b*) The variance of both $X$ and $Y$ is 2.

(*c*) The correlation coefficient of $X$ and $Y$ is $\frac{1}{2}$.

**3.53.** Consider a bivariate r.v. $(X, Y)$, where $X$ and $Y$ denote the horizontal and vertical miss distances, respectively, from a target when a bullet is fired. Assume that $X$ and $Y$ are independent and that the probability of the bullet landing on any point of the $xy$ plane depends only on the distance of the point from the target. Show that $(X, Y)$ is a bivariate normal r.v.

From the assumption, we have

$$f_{XY}(x, y) = f_X(x)f_Y(y) = g(x^2 + y^2) \tag{3.109}$$

for some function $g$. Differentiating Eq. (*3.109*) with respect to $x$, we have

$$f'_X(x)f_Y(y) = 2xg'(x^2 + y^2) \tag{3.110}$$

Dividing Eq. (*3.110*) by Eq. (*3.109*) and rearranging, we get

$$\frac{f'_X(x)}{2xf_X(x)} = \frac{g'(x^2 + y^2)}{g(x^2 + y^2)} \tag{3.111}$$

Note that the left-hand side of Eq. (3.111) depends only on $x$, whereas the right-hand side depends only on $x^2 + y^2$; thus

$$\frac{f'_X(x)}{xf_X(x)} = c \tag{3.112}$$

where $c$ is a constant. Rewriting Eq. (3.112) as

$$\frac{f'_X(x)}{f_X(x)} = cx \qquad \text{or} \qquad \frac{d}{dx}[\ln f_X(x)] = cx \tag{3.113}$$

and integrating both sides, we get

$$\ln f_X(x) = \frac{c}{2}x^2 + a \qquad \text{or} \qquad f_X(x) = ke^{cx^2/2}$$

where $a$ and $k$ are constants. By the properties of a pdf, the constant $c$ must be negative, and setting $c = -1/\sigma^2$, we have

$$f_X(x) = ke^{-x^2/(2\sigma^2)}$$

Thus, by Eq. (2.52), $X = N(0; \sigma^2)$ and

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-x^2/(2\sigma^2)}$$

In a similar way, we can obtain the pdf of $Y$ as

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-y^2/(2\sigma^2)}$$

Since $X$ and $Y$ are independent, the joint pdf of $(X, Y)$ is

$$f_{XY}(x, y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2 + y^2)/(2\sigma^2)}$$

which indicates that $(X, Y)$ is a bivariate normal r.v.

**3.54.** Let $(X_1, X_2, \ldots, X_n)$ be an $n$-variate normal r.v. with its joint pdf given by Eq. (3.92). Show that if the covariance of $X_i$ and $X_j$ is zero for $i \neq j$, that is,

$$\mathrm{Cov}(X_i, X_j) = \sigma_{ij} = \begin{cases} \sigma_i^2 & i = j \\ 0 & i \neq j \end{cases} \tag{3.114}$$

then $X_1, X_2, \ldots, X_n$ are independent.

From Eq. (3.94) with Eq. (3.114), the covariance matrix $K$ becomes

$$K = \begin{bmatrix} \sigma_1^2 & 0 & \cdots & 0 \\ 0 & \sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_n^2 \end{bmatrix} \tag{3.115}$$

It therefore follows that

$$|\det K|^{1/2} = \sigma_1\sigma_2 \cdots \sigma_n = \prod_{i=1}^{n} \sigma_i \tag{3.116}$$

and

$$K^{-1} = \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sigma_n^2} \end{bmatrix} \tag{3.117}$$

Then we can write

$$(\mathbf{x} - \boldsymbol{\mu})^T K^{-1} (\mathbf{x} - \boldsymbol{\mu}) = \sum_{i=1}^{n} \left( \frac{x_i - \mu_i}{\sigma_i} \right)^2 \tag{3.118}$$

Substituting Eqs. (*3.116*) and (*3.118*) into Eq. (*3.92*), we obtain

$$f_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2} \left( \prod_{i=1}^{n} \sigma_i \right)} \exp\left[ -\frac{1}{2} \sum_{i=1}^{n} \left( \frac{x_i - \mu_i}{\sigma_i} \right)^2 \right] \tag{3.119}$$

Now Eq. (*3.119*) can be rewritten as

$$f_{X_1 \cdots X_n}(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_{X_i}(x_i) \tag{3.120}$$

where

$$f_{X_i}(x_i) = \frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-(x_i - \mu_i)^2/(2\sigma_i^2)}$$

Thus we conclude that $X_1, X_2, \ldots, X_n$ are independent.

## Supplementary Problems

**3.55.** Consider an experiment of tossing a fair coin three times. Let $(X, Y)$ be a bivariate r.v., where $X$ denotes the number of heads on the first two tosses and $Y$ denotes the number of heads on the third toss.

(*a*) Find the range of $X$.

(*b*) Find the range of $Y$.

(*c*) Find the range of $(X, Y)$.

(*d*) Find (i) $P(X \leq 2, Y \leq 1)$; (ii) $P(X \leq 1, Y \leq 1)$; and (iii) $P(X \leq 0, Y \leq 0)$.

*Ans.* (*a*) $R_X = \{0, 1, 2\}$
(*b*) $R_Y = \{0, 1\}$
(*c*) $R_{XY} = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$
(*d*) (i) $P(X \leq 2, Y \leq 1) = 1$; (ii) $P(X \leq 1, Y \leq 1) = \frac{3}{4}$; and (iii) $P(X \leq 0, Y \leq 0) = \frac{1}{8}$

**3.56.** Let $F_{XY}(x, y)$ be a joint cdf of a bivariate r.v. $(X, Y)$. Show that

$$P(X > a, Y > c) = 1 - F_X(a) - F_Y(c) + F_{XY}(a, c)$$

where $F_X(x)$ and $F_Y(y)$ are marginal cdf's of $X$ and $Y$, respectively.

*Hint:* Set $x_1 = a$, $y_1 = c$, and $x_2 = y_2 = \infty$ in Eq. (*3.95*) and use Eqs. (*3.13*) and (*3.14*).

**3.57.** Let the joint pmf of $(X, Y)$ be given by

$$p_{XY}(x_i, y_j) = \begin{cases} k(x_i + y_j) & x_i = 1, 2, 3;\ y_j = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the marginal pmf's of $X$ and $Y$.

*Ans.* (*a*) $k = \frac{1}{21}$

(*b*) $p_X(x_i) = \frac{1}{21}(2x_i + 3) \qquad x_i = 1, 2, 3$

$p_Y(y_j) = \frac{1}{21}(6 + 3y_j) \qquad y_j = 1, 2$

**3.58.** The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} ke^{-(x+2y)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find $P(X > 1, Y < 1)$, $P(X < Y)$, and $P(X \leq 2)$.

*Ans.* (*a*) $k = 2$

(*b*) $P(X > 1, Y < 1) = e^{-1} - e^{-3} \approx 0.318$; $P(X < Y) = \frac{1}{3}$; $P(X \leq 2) = 1 - e^{-2} \approx 0.865$

**3.59.** Let $(X, Y)$ be a bivariate r.v., where $X$ is a uniform r.v. over (0, 0.2) and $Y$ is an exponential r.v. with parameter 5, and $X$ and $Y$ are independent.

(*a*) Find the joint pdf of $(X, Y)$.

(*b*) Find $P(Y \leq X)$.

*Ans.* (*a*) $f_{XY}(x, y) = \begin{cases} 25e^{-(5y)} & 0 < x < 0.2,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$

(*b*) $P(Y \leq X) = e^{-1} \approx 0.368$

**3.60.** Let the joint pdf of $(X, Y)$ be given by

$$f_{XY}(x, y) = \begin{cases} xe^{-x(y+1)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Show that $f_{XY}(x, y)$ satisfies Eq. (*3.26*).

(*b*) Find the marginal pdf's of X and Y.

*Ans.* (*b*) $f_X(x) = e^{-x} \qquad x > 0$

$f_Y(y) = \dfrac{1}{(y+1)^2} \qquad y > 0$

**3.61.** The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} kx^2(4 - y) & x < y < 2x,\ 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a constant.

(*a*) Find the value of $k$.

(*b*) Find the marginal pdf's of $X$ and $Y$.

*Ans.* (*a*) $k = \frac{5}{32}$

(*b*) $f_X(x) = \frac{5}{32}x^3(4 - \frac{3}{2}x) \qquad 0 < x < 2$

$$f_Y(y) = \begin{cases} (\frac{5}{32})\frac{7}{24}(4 - y)y^3 & 0 < y < 2 \\ (\frac{5}{32})\frac{1}{3}(4 - y)(8 - \frac{1}{8}y^3) & 2 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

**3.62.** The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} xye^{-(x^2+y^2)/2} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Find the marginal pdf's of $X$ and $Y$.

(*b*) Are $X$ and $Y$ independent?

*Ans.* (*a*) $f_X(x) = xe^{-x^2/2} \qquad x > 0$

$f_Y(y) = ye^{-y^2/2} \qquad y > 0$

(*b*) Yes

**3.63.** The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Are $X$ and $Y$ independent?

(*b*) Find the conditional pdf's of $X$ and $Y$.

*Ans.* (*a*) Yes

(*b*) $f_{X|Y}(x \mid y) = e^{-x} \qquad x > 0$

$f_{Y|X}(y \mid x) = e^{-y} \qquad y > 0$

**3.64.** The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x, y) = \begin{cases} e^{-y} & 0 < x \le y \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Find the conditional pdf's of $Y$, given that $X = x$.

(*b*) Find the conditional cdf's of $Y$, given that $X = x$.

*Ans.* (*a*) $f_{Y|X}(y \mid x) = e^{x-y} \qquad y \ge x$

(*b*) $F_{Y|X}(y \mid x) = \begin{cases} 0 & y \le x \\ 1 - e^{x-y} & y \ge x \end{cases}$

**3.65.** Consider the bivariate r.v. $(X, Y)$ of Prob. 3.14.

(*a*) Find the mean and the variance of $X$.

(*b*) Find the mean and the variance of $Y$.

(*c*) Find the covariance of $X$ and $Y$.

(*d*) Find the correlation coefficient of $X$ and $Y$.

*Ans.* (*a*) $E(X) = \frac{29}{18}$, $\mathrm{Var}(X) = \frac{77}{324}$

(*b*) $E(Y) = \frac{14}{9}$, $\mathrm{Var}(Y) = \frac{20}{81}$

(*c*) $\mathrm{Cov}(X, Y) = -\frac{1}{162}$

(*d*) $\rho = -0.025$

**3.66.** Consider a bivariate r.v. $(X, Y)$ with joint pdf

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/(2\sigma^2)} \qquad -\infty < x, y < \infty$$

Find $P[(X, Y) \mid x^2 + y^2 \le a^2]$.

*Ans.* $1 - e^{-a^2/(2\sigma^2)}$

**3.67.** Let $(X, Y)$ be a bivariate normal r.v., where $X$ and $Y$ each have zero mean and variance $\sigma^2$, and the correlation coefficient of $X$ and $Y$ is $\rho$. Find the joint pdf of $(X, Y)$.

*Ans.* $f_{XY}(x, y) = \dfrac{1}{2\pi\sigma^2(1-\rho^2)^{1/2}} \exp\left[-\dfrac{1}{2}\dfrac{x^2 - 2\rho xy + y^2}{\sigma^2(1-\rho^2)}\right]$

**3.68.** The joint pdf of a bivariate r.v. $(X, Y)$ is given by

$$f_{XY}(x, y) = \frac{1}{\sqrt{3}\pi} \exp\left[-\tfrac{2}{3}(x^2 - xy + y^2)\right]$$

(*a*) Find the means and variances of $X$ and $Y$.

(*b*) Find the correlation coefficient of $X$ and $Y$.

*Ans.* (*a*) $\mu_X = \mu_Y = 0 \qquad \sigma_X^2 = \sigma_Y^2 = 1$

(*b*) $\rho = \frac{1}{2}$

**3.69.** Let $(X, Y, Z)$ be a trivariate r.v., where $X$, $Y$, and $Z$ are independent and each has a uniform distribution over (0, 1). Compute $P(X \geq Y \geq Z)$.

*Ans.* $\frac{1}{6}$

# Chapter 4

# Functions of Random Variables, Expectation, Limit Theorems

## 4.1 INTRODUCTION

In this chapter we study a few basic concepts of functions of random variables and investigate the expected value of a certain function of a random variable. The techniques of moment generating functions and characteristic functions, which are very useful in some applications, are presented. Finally, the laws of large numbers and the central limit theorem, which is one of the most remarkable results in probability theory, are discussed.

## 4.2 FUNCTIONS OF ONE RANDOM VARIABLE

### A. Random Variable $g(X)$:

Given a r.v. $X$ and a function $g(x)$, the expression

$$Y = g(X) \tag{4.1}$$

defines a new r.v. $Y$. With $y$ a given number, we denote $D_Y$ the subset of $R_X$ (range of $X$) such that $g(x) \leq y$. Then

$$(Y \leq y) = [g(X) \leq y] = (X \in D_Y) \tag{4.2}$$

where $(X \in D_Y)$ is the event consisting of all outcomes $\zeta$ such that the point $X(\zeta) \in D_Y$. Hence

$$F_Y(y) = P(Y \leq y) = P[g(X) \leq y] = P(X \in D_Y) \tag{4.3}$$

If $X$ is a continuous r.v. with pdf $f_X(x)$, then

$$F_Y(y) = \int_{D_Y} f_X(x)\, dx \tag{4.4}$$

### B. Determination of $f_Y(y)$ from $f_X(x)$:

Let $X$ be a continuous r.v. with pdf $f_X(x)$. If the transformation $y = g(x)$ is one-to-one and has the inverse transformation

$$x = g^{-1}(y) = h(y) \tag{4.5}$$

then the pdf of $Y$ is given by (Prob. 4.2)

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = f_X[h(y)] \left| \frac{dh(y)}{dy} \right| \tag{4.6}$$

Note that if $g(x)$ is a continuous monotonic increasing or decreasing function, then the transformation $y = g(x)$ is one-to-one. If the transformation $y = g(x)$ is not one-to-one, $f_Y(y)$ is obtained as follows: Denoting the real roots of $y = g(x)$ by $x_k$, that is,

$$y = g(x_1) = \cdots = g(x_k) = \cdots \tag{4.7}$$

then
$$f_Y(y) = \sum_k \frac{f_X(x_k)}{|g'(x_k)|} \tag{4.8}$$

where $g'(x)$ is the derivative of $g(x)$.

## 4.3. FUNCTIONS OF TWO RANDOM VARIABLES

### A. One Function of Two Random Variables:

Given two r.v.'s $X$ and $Y$ and a function $g(x, y)$, the expression
$$Z = g(X, Y) \tag{4.9}$$

defines a new r.v. $Z$. With $z$ a given number, we denote $D_Z$ the subset of $R_{XY}$ [range of $(X, Y)$] such that $g(x, y) \leq z$. Then
$$(Z \leq z) = [g(X, Y) \leq z] = \{(X, Y) \in D_Z\} \tag{4.10}$$

where $\{(X, Y) \in D_Z\}$ is the event consisting of all outcomes $\zeta$ such that the point $\{X(\zeta), Y(\zeta)\} \in D_Z$. Hence
$$F_Z(z) = P(Z \leq z) = P[g(X, Y) \leq z] = P\{(X, Y) \in D_Z\} \tag{4.11}$$

If $X$ and $Y$ are continuous r.v.'s with joint pdf $f_{XY}(x, y)$, then
$$F_Z(z) = \iint_{D_Z} f_{XY}(x, y)\, dx\, dy \tag{4.12}$$

### B. Two Functions of Two Random Variables:

Given two r.v.'s $X$ and $Y$ and two functions $g(x, y)$ and $h(x, y)$, the expression
$$Z = g(X, Y) \qquad W = h(X, Y) \tag{4.13}$$

defines two new r.v.'s $Z$ and $W$. With $z$ and $w$ two given numbers, we denote $D_{ZW}$ the subset of $R_{XY}$ [range of $(X, Y)$] such that $g(x, y) \leq z$ and $h(x, y) \leq w$. Then
$$(Z \leq z, W \leq w) = [g(X, Y) \leq z, h(X, Y) \leq w] = \{(X, Y) \in D_{ZW}\} \tag{4.14}$$

where $\{(X, Y) \in D_{ZW}\}$ is the event consisting of all outcomes $\zeta$ such that the point $\{X(\zeta), Y(\zeta)\} \in D_{ZW}$. Hence
$$\begin{aligned} F_{ZW}(z, w) = P(Z \leq z, W \leq w) &= P[g(X, Y) \leq z, h(X, Y) \leq w] \\ &= P\{(X, Y) \in D_{ZW}\} \end{aligned} \tag{4.15}$$

In the continuous case, we have
$$F_{ZW}(z, w) = \iint_{D_{ZW}} f_{XY}(x, y)\, dx\, dy \tag{4.16}$$

***Determination of $f_{ZW}(z, w)$ from $f_{XY}(x, y)$:***

Let $X$ and $Y$ be two continuous r.v.'s with joint pdf $f_{XY}(x, y)$. If the transformation
$$z = g(x, y) \qquad w = h(x, y) \tag{4.17}$$

is one-to-one and has the inverse transformation
$$x = q(z, w) \qquad y = r(z, w) \tag{4.18}$$

then the joint pdf of $Z$ and $W$ is given by

$$f_{ZW}(z, w) = f_{XY}(x, y)\,|J(x, y)|^{-1} \tag{4.19}$$

where $x = q(z, w)$, $y = r(z, w)$, and

$$J(x, y) = \begin{vmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} \tag{4.20}$$

which is the jacobian of the transformation (*4.17*). If we define

$$\bar{J}(z, w) = \begin{vmatrix} \dfrac{\partial q}{\partial z} & \dfrac{\partial q}{\partial w} \\ \dfrac{\partial r}{\partial z} & \dfrac{\partial r}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} \tag{4.21}$$

then

$$|\bar{J}(z, w)| = |J(x, y)|^{-1} \tag{4.22}$$

and Eq. (*4.19*) can be expressed as

$$f_{ZW}(z, w) = f_{XY}[q(z, w), r(z, w)]\,|\bar{J}(z, w)| \tag{4.23}$$

## 4.4 FUNCTIONS OF $n$ RANDOM VARIABLES

### A. One Function of $n$ Random Variables:

Given $n$ r.v.'s $X_1, \ldots, X_n$ and a function $g(x_1, \ldots, x_n)$, the expression

$$Y = g(X_1, \ldots, X_n) \tag{4.24}$$

defines a new r.v. $Y$. Then

$$(Y \le y) = [g(X_1, \ldots, X_n) \le y] = [(X_1, \ldots, X_n) \in D_Y] \tag{4.25}$$

and

$$F_Y(y) = P[g(X_1, \ldots, X_n) \le y] = P[(X_1, \ldots, X_n) \in D_Y] \tag{4.26}$$

where $D_Y$ is the subset of the range of $(X_1, \ldots, X_n)$ such that $g(x_1, \ldots, x_n) \le y$. If $X_1, \ldots, X_n$ are continuous r.v.'s with joint pdf $f_{X_1 \cdots X_n}(x_1, \ldots, x_n)$, then

$$F_Y(y) = \int_{D_Y} \cdots \int f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \tag{4.27}$$

### B. $n$ Functions of $n$ Random Variables:

When the joint pdf of $n$ r.v.'s $X_1, \ldots, X_n$ is given and we want to determine the joint pdf of $n$ r.v.'s $Y_1, \ldots, Y_n$, where

$$\begin{aligned} Y_1 &= g_1(X_1, \ldots, X_n) \\ &\vdots \\ Y_n &= g_n(X_1, \ldots, X_n) \end{aligned} \tag{4.28}$$

the approach is the same as for two r.v.'s. We shall assume that the transformation

$$\begin{aligned} y_1 &= g_1(x_1, \ldots, x_n) \\ &\vdots \\ y_n &= g_n(x_1, \ldots, x_n) \end{aligned} \tag{4.29}$$

is one-to-one and has the inverse transformation

$$\begin{aligned} x_1 &= h_1(y_1, \ldots, y_n) \\ &\vdots \\ x_n &= h_n(y_1, \ldots, y_n) \end{aligned} \tag{4.30}$$

Then the joint pdf of $Y_1, \ldots, Y_n$ is given by

$$f_{Y_1 \cdots Y_n}(y_1, \ldots, y_n) = f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\,|J(x_1, \ldots, x_n)|^{-1} \tag{4.31}$$

where

$$J(x_1, \ldots, x_n) = \begin{vmatrix} \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial x_1} & \cdots & \dfrac{\partial g_n}{\partial x_n} \end{vmatrix} \tag{4.32}$$

which is the jacobian of the transformation (4.29).

## 4.5 EXPECTATION

### A. Expectation of a Function of One Random Variable:

The expectation of $Y = g(X)$ is given by

$$E(Y) = E[g(X)] = \begin{cases} \displaystyle\sum_i g(x_i) p_X(x_i) & \text{(discrete case)} \\ \displaystyle\int_{-\infty}^{\infty} g(x) f_X(x)\, dx & \text{(continuous case)} \end{cases} \tag{4.33}$$

### B. Expectation of a Function of More than One Random Variable:

Let $X_1, \ldots, X_n$ be $n$ r.v.'s, and let $Y = g(X_1, \ldots, X_n)$. Then

$$E(Y) = E[g(X)] = \begin{cases} \displaystyle\sum_{x_1} \cdots \sum_{x_n} g(x_1, \ldots, x_n) p_{X_1 \cdots X_n}(x_1, \ldots, x_n) & \text{(discrete case)} \\ \displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \ldots, x_n) f_{X_1 \cdots X_n}(x_1, \ldots, x_n)\, dx_1 \cdots dx_n & \text{(continuous case)} \end{cases} \tag{4.34}$$

### C. Linearity Property of Expectation:

Note that the expectation operation is linear (Prob. 4.39), and we have

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) \tag{4.35}$$

where $a_i$'s are constants. If r.v.'s $X$ and $Y$ are independent, then we have (Prob. 4.41)

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)] \tag{4.36}$$

The relation (4.36) can be generalized to a mutually independent set of $n$ r.v.'s $X_1, \ldots, X_n$:

$$E\left[\prod_{i=1}^{n} g_i(X_i)\right] = \prod_{i=1}^{n} E[g_i(X_i)] \tag{4.37}$$

### D. Conditional Expectation as a Random Variable:

In Sec. 3.8 we defined the conditional expectation of $Y$ given $X = x$, $E(Y \mid x)$ [Eq. (*3.58*)], which is, in general, a function of $x$, say $H(x)$. Now $H(X)$ is a function of the r.v. $X$; that is,

$$H(X) = E(Y \mid X) \tag{4.38}$$

Thus, $E(Y \mid X)$ is a function of the r.v. $X$. Note that $E(Y \mid X)$ has the following property (Prob. 4.38):

$$E[E(Y \mid X)] = E(Y) \tag{4.39}$$

## 4.6 MOMENT GENERATING FUNCTIONS

### A. Definition:

The *moment generating function* of a r.v. $X$ is defined by

$$M_X(t) = E(e^{tX}) = \begin{cases} \sum_i e^{tx_i} p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx & \text{(continuous case)} \end{cases} \tag{4.40}$$

where $t$ is a real variable. Note that $M_X(t)$ may not exist for all r.v.'s $X$. In general, $M_X(t)$ will exist only for those values of $t$ for which the sum or integral of Eq. (*4.40*) converges absolutely. Suppose that $M_X(t)$ exists. If we express $e^{tX}$ formally and take expectation, then

$$\begin{aligned} M_X(t) = E(e^{tX}) &= E\left[1 + tX + \frac{1}{2!}(tX)^2 + \cdots + \frac{1}{k!}(tX)^k + \cdots\right] \\ &= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^k}{k!}E(X^k) + \cdots \end{aligned} \tag{4.41}$$

and the $k$th moment of $X$ is given by

$$m_k = E(X^k) = M_X^{(k)}(0) \qquad k = 1, 2, \ldots \tag{4.42}$$

where

$$M_X^{(k)}(0) = \frac{d^k}{dt^k} M_X(t)\bigg|_{t=0} \tag{4.43}$$

### B. Joint Moment Generating Function:

The joint moment generating function $M_{XY}(t_1, t_2)$ of two r.v.'s $X$ and $Y$ is defined by

$$M_{XY}(t_1, t_2) = E[e^{(t_1 X + t_2 Y)}] \tag{4.44}$$

where $t_1$ and $t_2$ are real variables. Proceeding as we did in Eq. (*4.41*), we can establish that

$$M_{XY}(t_1, t_2) = E[e^{(t_1 X + t_2 Y)}] = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{t_1^k t_2^n}{k!\, n!} E(X^k Y^n) \tag{4.45}$$

and the $(k, n)$ joint moment of $X$ and $Y$ is given by

$$m_{kn} = E(X^k Y^n) = M_{XY}^{(kn)}(0, 0) \tag{4.46}$$

where

$$M_{XY}^{(kn)}(0, 0) = \frac{\partial^{k+n}}{\partial^k t_1 \partial^n t_2} M_{XY}(t_1, t_2)\bigg|_{t_1 = t_2 = 0} \tag{4.47}$$

In a similar fashion, we can define the joint moment generating function of $n$ r.v.'s $X_1, \ldots, X_n$ by

$$M_{X_1 \cdots X_n}(t_1, \ldots, t_n) = E[e^{(t_1 X_1 + \cdots + t_n X_n)}] \tag{4.48}$$

from which the various moments can be computed. If $X_1, \ldots, X_n$ are independent, then

$$M_{X_1 \cdots X_n}(t_1, \ldots, t_n) = E[e^{(t_1 X_1 + \cdots + t_n X_n)}] = E(e^{t_1 X_1} \cdots e^{t_n X_n})$$
$$= E(e^{t_1 X_1}) \cdots E(e^{t_n X_n}) = M_{X_1}(t_1) \cdots M_{X_n}(t_n) \tag{4.49}$$

### C. Lemmas for Moment Generating Functions:

Two important lemmas concerning moment generating functions are stated in the following:

**Lemma 4.1:** If two r.v.'s have the same moment generating functions, then they must have the same distribution.

**Lemma 4.2:** Given cdf's $F(x)$, $F_1(x)$, $F_2(x)$, ... with corresponding moment generating functions $M(t)$, $M_1(t)$, $M_2(t), \ldots$, then $F_n(x) \to F(x)$ if $M_n(t) \to M(t)$.

## 4.7 CHARACTERISTIC FUNCTIONS

### A. Definition:

The *characteristic function* of a r.v. $X$ is defined by

$$\Psi_X(\omega) = E(e^{j\omega X}) = \begin{cases} \sum_i e^{j\omega x_i} p_X(x_i) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx & \text{(continuous case)} \end{cases} \tag{4.50}$$

where $\omega$ is a real variable and $j = \sqrt{-1}$. Note that $\Psi_X(\omega)$ is obtained by replacing $t$ in $M_X(t)$ by $j\omega$ if $M_X(t)$ exists. Thus, the characteristic function has all the properties of the moment generating function. Now

$$|\Psi_X(\omega)| = \left| \sum_i e^{j\omega x_i} p_X(x_i) \right| \le \sum_i |e^{j\omega x_i} p_X(x_i)| = \sum_i p_X(x_i) = 1 < \infty$$

for the discrete case and

$$|\Psi_X(\omega)| = \left| \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx \right| \le \int_{-\infty}^{\infty} |e^{j\omega x} f_X(x)\, dx| = \int_{-\infty}^{\infty} f_X(x)\, dx = 1 < \infty$$

for the continuous case. Thus, the characteristic function $\Psi_X(\omega)$ is always defined even if the moment function $M_X(t)$ is not (Prob. 4.58). Note that $\Psi_X(\omega)$ of Eq. (*4.50*) for the continuous case is the Fourier transform (with the sign of $j$ reversed) of $f_X(x)$. Because of this fact, if $\Psi_X(\omega)$ is known, $f_X(x)$ can be found from the inverse Fourier transform; that is,

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi_X(\omega) e^{-j\omega x}\, d\omega \tag{4.51}$$

### B. Joint Characteristic Functions:

The joint characteristic function $\Psi_{XY}(\omega_1, \omega_2)$ of two r.v.'s $X$ and $Y$ is defined by

$$\Psi_{XY}(\omega_1, \omega_2) = E[e^{j(\omega_1 X + \omega_2 Y)}]$$
$$= \begin{cases} \sum_i \sum_k e^{j(\omega_1 x_i + \omega_2 y_k)} p_{XY}(x_i, y_k) & \text{(discrete case)} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{j(\omega_1 x + \omega_2 y)} f_{XY}(x, y)\, dx\, dy & \text{(continuous case)} \end{cases} \tag{4.52}$$

where $\omega_1$ and $\omega_2$ are real variables.

The expression of Eq. (4.52) for the continuous case is recognized as the two-dimensional Fourier transform (with the sign of $j$ reversed) of $f_{XY}(x, y)$. Thus, from the inverse Fourier transform, we have

$$f_{XY}(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{XY}(\omega_1, \omega_2) e^{-j(\omega_1 x + \omega_2 y)} \, d\omega_1 \, d\omega_2 \tag{4.53}$$

From Eqs. (4.50) and (4.52), we see that

$$\Psi_X(\omega) = \Psi_{XY}(\omega, 0) \qquad \Psi_Y(\omega) = \Psi_{XY}(0, \omega) \tag{4.54}$$

which are called *marginal characteristic functions.*

Similarly, we can define the joint characteristic function of $n$ r.v.'s $X_1, \ldots, X_n$ by

$$\Psi_{X_1 \cdots X_n}(\omega_1, \ldots, \omega_n) = E[e^{j(\omega_1 X_1 + \cdots + \omega_n X_n)}] \tag{4.55}$$

As in the case of the moment generating function, if $X_1, \ldots, X_n$ are independent, then

$$\Psi_{X_1 \cdots X_n}(\omega_1, \ldots, \omega_n) = \Psi_{X_1}(\omega_1) \cdots \Psi_{X_n}(\omega_n) \tag{4.56}$$

### C. Lemmas for Characteristic Functions:

As with the moment generating function, we have the following two lemmas:

**Lemma 4.3:** A distribution function is uniquely determined by its characteristic function.

**Lemma 4.4:** Given cdf's $F(x)$, $F_1(x)$, $F_2(x)$, ... with corresponding characteristic functions $\Psi(\omega)$, $\Psi_1(\omega)$, $\Psi_2(\omega)$, ..., then $F_n(x) \to F(x)$ at points of continuity of $F(x)$ if and only if $\Psi_n(\omega) \to \Psi(\omega)$ for every $\omega$.

## 4.8 THE LAWS OF LARGE NUMBERS AND THE CENTRAL LIMIT THEOREM

### A. The Weak Law of Large Numbers:

Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed r.v.'s each with a finite mean $E(X_i) = \mu$. Let

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i = \frac{1}{n} (X_1 + \cdots + X_n) \tag{4.57}$$

Then, for any $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0 \tag{4.58}$$

Equation (4.58) is known as the *weak law of large numbers*, and $\bar{X}_n$ is known as the *sample mean.*

### B. The Strong Law of Large Numbers:

Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed r.v.'s each with a finite mean $E(X_i) = \mu$. Then, for any $\varepsilon > 0$,

$$P\left( \lim_{n \to \infty} |\bar{X}_n - \mu| > \varepsilon \right) = 0 \tag{4.59}$$

where $\bar{X}_n$ is the sample mean defined by Eq. (4.57). Equation (4.59) is known as the *strong law of large numbers.*

Notice the important difference between Eqs. (*4.58*) ånd (*4.59*). Equation (*4.58*) tells us how a sequence of probabilities converges, and Eq. (*4.59*) tells us how the sequence of r.v.'s behaves in the limit. The strong law of large numbers tells us that the sequence $(\bar{X}_n)$ is converging to the contant $\mu$.

### C. The Central Limit Theorem:

The *central limit theorem* is one of the most remarkable results in probability theory. There are many versions of this theorem. In its simplest form, the central limit theorem is stated as follows:

Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed r.v.'s each with mean $\mu$ and variance $\sigma^2$. Let

$$Z_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \tag{4.60}$$

where $\bar{X}_n$ is defined by Eq. (*4.57*). Then the distribution of $Z_n$ tends to the standard normal as $n \to \infty$; that is,

$$\lim_{n\to\infty} Z_n = N(0; 1) \tag{4.61}$$

or

$$\lim_{n\to\infty} F_{Z_n}(z) = \lim_{n\to\infty} P(Z_n \leq z) = \Phi(z) \tag{4.62}$$

where $\Phi(z)$ is the cdf of a standard normal r.v. [Eq. (*2.54*)]. Thus, the central limit theorem tells us that for large $n$, the distribution of the sum $S_n = X_1 + \cdots + X_n$ is approximately normal regardless of the form of the distribution of the individual $X_i$'s. Notice how much stronger this theorem is than the laws of large numbers. In practice, whenever an observed r.v. is known to be a sum of a large number of r.v.'s, then the central limit theorem gives us some justification for assuming that this sum is normally distributed.

## Solved Problems

### FUNCTIONS OF ONE RANDOM VARIABLE

**4.1.** If $X$ is $N(\mu; \sigma^2)$, then show that $Z = (X - \mu)/\sigma$ is a standard normal r.v.; that is, $N(0; 1)$.

The cdf of $Z$ is

$$F_Z(z) = P(Z \leq z) = P\left(\frac{X - \mu}{\sigma} \leq z\right) = P(X \leq z\sigma + \mu)$$
$$= \int_{-\infty}^{z\sigma+\mu} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

By the change of variable $y = (x - \mu)/\sigma$ (that is, $x = \sigma y + \mu$), we obtain

$$F_Z(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy$$

and $$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

which indicates that $Z = N(0; 1)$.

**4.2.** Verify Eq. (*4.6*).

Assume that $y = g(x)$ is a continuous monotonically increasing function [Fig. 4-1(*a*)]. Since $y = g(x)$ is monotonically increasing, it has an inverse that we denote by $x = g^{-1}(y) = h(y)$. Then

$$F_Y(y) = P(Y \leq y) = P[X \leq h(y)] = F_X[h(y)] \tag{4.63}$$

and $$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \{F_X[h(y)]\}$$

Applying the chain rule of differentiation to this expression yields

$$f_Y(y) = f_X[h(y)] \frac{d}{dy} h(y)$$

which can be written as

$$f_Y(y) = f_X(x) \frac{dx}{dy} \qquad x = h(y) \tag{4.64}$$

If $y = g(x)$ is monotonically decreasing [Fig. 4.1(*b*)], then

$$F_Y(y) = P(Y \leq y) = P[X > h(y)] = 1 - F_X[h(y)] \tag{4.65}$$

Thus, $$f_Y(y) = \frac{d}{dy} F_Y(y) = -f_X(x) \frac{dx}{dy} \qquad x = h(y) \tag{4.66}$$

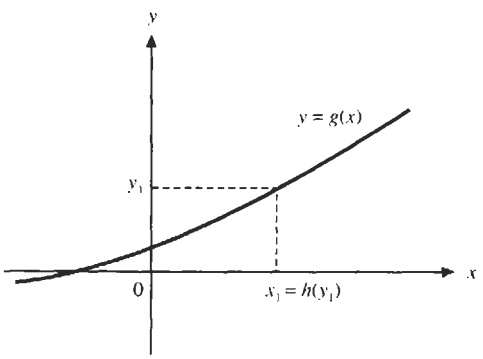
In Eq. (*4.66*), since $y = g(x)$ is monotonically decreasing, $dy/dx$ (and $dx/dy$) is negative. Combining Eqs. (*4.64*) and (*4.66*), we obtain

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right| = f_X[h(y)] \left| \frac{dh(y)}{dy} \right|$$

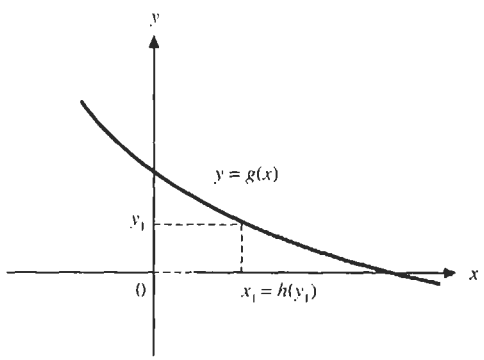
which is valid for any continuous monotonic (increasing or decreasing) function $y = g(x)$.

**4.3.** Let $X$ be a r.v. with cdf $F_X(x)$ and pdf $f_X(x)$. Let $Y = aX + b$, where $a$ and $b$ are real constants and $a \neq 0$.

(*a*) Find the cdf of $Y$ in terms of $F_X(x)$.




(*a*)




(*b*)

**Fig. 4-1**

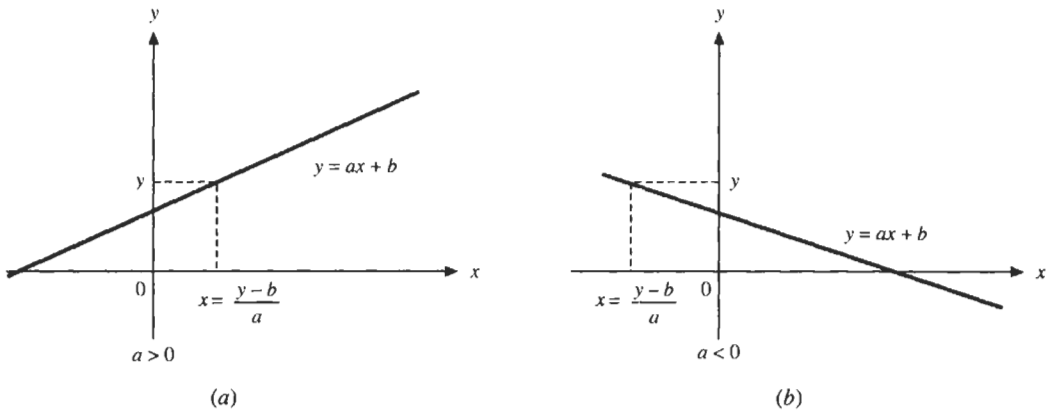



**Fig. 4-2**

(*b*) Find the pdf of $Y$ in terms of $f_X(x)$.

(*a*) If $a > 0$, then [Fig. 4-2(*a*)]

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = P\left(X \le \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right) \tag{4.67}$$

If $a < 0$, then [Fig. 4-2(*b*)]

$$\begin{aligned} F_Y(y) &= P(Y \le y) = P(aX + b \le y) = P(aX \le y - b) \\ &= P\left(X \ge \frac{y-b}{a}\right) \quad \text{(since } a < 0\text{, note the change in the inequality sign)} \\ &= 1 - P\left(X < \frac{y-b}{a}\right) \\ &= 1 - P\left(X \le \frac{y-b}{a}\right) + P\left(X = \frac{y-b}{a}\right) \\ &= 1 - F_X\left(\frac{y-b}{a}\right) + P\left(X = \frac{y-b}{a}\right) \end{aligned} \tag{4.68}$$

Note that if $X$ is continuous, then $P[X = (y - b)/a] = 0$, and

$$F_Y(y) = 1 - F_X\left(\frac{y-b}{a}\right) \qquad a < 0 \tag{4.69}$$

(*b*) From Fig. 4-2, we see that $y = g(x) = ax + b$ is a continuous monotonically increasing ($a > 0$) or decreasing ($a < 0$) function. Its inverse is $x = g^{-1}(y) = h(y) = (y - b)/a$, and $dx/dy = 1/a$. Thus, by Eq. (*4.6*),

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \tag{4.70}$$

Note that Eq. (*4.70*) can also be obtained by differentiating Eqs. (*4.67*) and (*4.69*) with respect to $y$.

**4.4.** Let $Y = aX + b$. Determine the pdf of $Y$, if $X$ is a uniform r.v. over (0, 1).

The pdf of $X$ is [Eq. (*2.44*)]

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then by Eq. (*4.70*), we get

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) = \begin{cases} \dfrac{1}{|a|} & y \in R_Y \\ 0 & \text{otherwise} \end{cases} \tag{4.71}$$

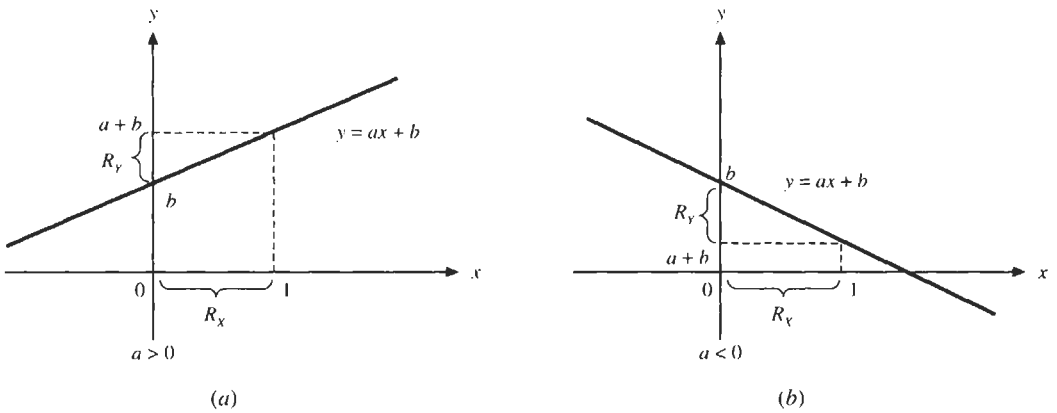



**Fig. 4-3**

The range $R_Y$ is found as follows: From Fig. 4-3, we see that

For $a > 0$: $\quad R_Y = \{y: b < y < a + b\}$

For $a < 0$: $\quad R_Y = \{y: a + b < y < b\}$

**4.5.** Let $Y = aX + b$. Show that if $X = N(\mu; \sigma^2)$, then $Y = N(a\mu + b; a^2\sigma^2)$, and find the values of $a$ and $b$ so that $Y = N(0; 1)$.

Since $X = N(\mu; \sigma^2)$, by Eq. (2.52),

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right]$$

Hence, by Eq. (4.70),

$$\begin{aligned} f_Y(y) &= \frac{1}{\sqrt{2\pi}\,|a|\,\sigma} \exp\left\{-\frac{1}{2\sigma^2}\left[\left(\frac{y - b}{a}\right) - \mu\right]^2\right\} \\ &= \frac{1}{\sqrt{2\pi}\,|a|\,\sigma} \exp\left\{-\frac{1}{2a^2\sigma^2}[y - (a\mu + b)]^2\right\} \end{aligned} \tag{4.72}$$

which is the pdf of $N(a\mu + b; a^2\sigma^2)$. Hence, $Y = N(a\mu + b; a^2\sigma^2)$. Next, let $a\mu + b = 0$ and $a^2\sigma^2 = 1$, from which we get $a = 1/\sigma$ and $b = -\mu/\sigma$. Thus, $Y = (X - \mu)/\sigma$ is $N(0; 1)$ (see Prob. 4.1).

**4.6.** Let $X$ be a r.v. with pdf $f_X(x)$. Let $Y = X^2$. Find the pdf of $Y$.

The event $A = (Y \le y)$ in $R_Y$ is equivalent to the event $B = (-\sqrt{y} \le X \le \sqrt{y})$ in $R_X$ (Fig. 4-4). If $y \le 0$, then

$$F_Y(y) = P(Y \le y) = 0$$

and $f_Y(y) = 0$. If $y > 0$, then

$$F_Y(y) = P(Y \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \tag{4.73}$$

and

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(\sqrt{y}) - \frac{d}{dy}F_X(-\sqrt{y}) = \frac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$

Thus,

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}[f_X(\sqrt{y}) + f_X(-\sqrt{y})] & y > 0 \\ 0 & y \le 0 \end{cases} \tag{4.74}$$

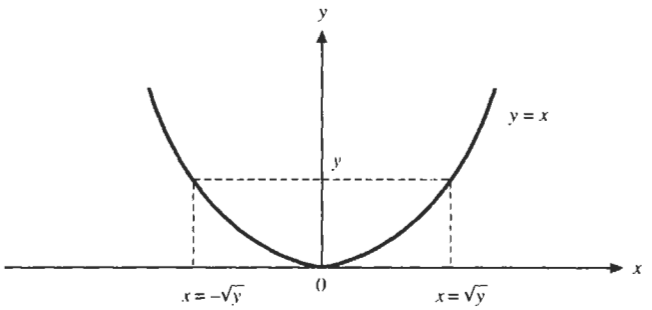



**Fig. 4-4**

*Alternative Solution:*

If $y < 0$, then the equation $y = x^2$ has no real solutions; hence $f_Y(y) = 0$. If $y > 0$, then $y = x^2$ has two solutions, $x_1 = \sqrt{y}$ and $x_2 = -\sqrt{y}$. Now, $y = g(x) = x^2$ and $g'(x) = 2x$. Hence, by Eq. (*4.8*),

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] & y > 0 \\ 0 & y < 0 \end{cases}$$

**4.7.** Let $Y = X^2$. Find the pdf of $Y$ if $X = N(0; 1)$.

Since $X = N(0; 1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Since $f_X(x)$ is an even function, by Eq. (*4.74*), we obtain

$$f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{y}} f_X(\sqrt{y}) = \dfrac{1}{\sqrt{2\pi y}} e^{-y/2} & y > 0 \\ 0 & y < 0 \end{cases} \tag{4.75}$$

**4.8.** Let $Y = X^2$. Find and sketch the pdf of $Y$ if $X$ is a uniform r.v. over $(-1, 2)$.

The pdf of $X$ is [Eq. (*2.44*)] [Fig. 4-5(*a*)]

$$f_X(x) = \begin{cases} \frac{1}{3} & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

In this case, the range of $Y$ is $(0, 4)$, and we must be careful in applying Eq. (*4.74*). When $0 < y < 1$, both $\sqrt{y}$ and $-\sqrt{y}$ are in $R_X = (-1, 2)$, and by Eq. (*4.74*),

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{3} + \frac{1}{3}\right) = \frac{1}{3\sqrt{y}}$$

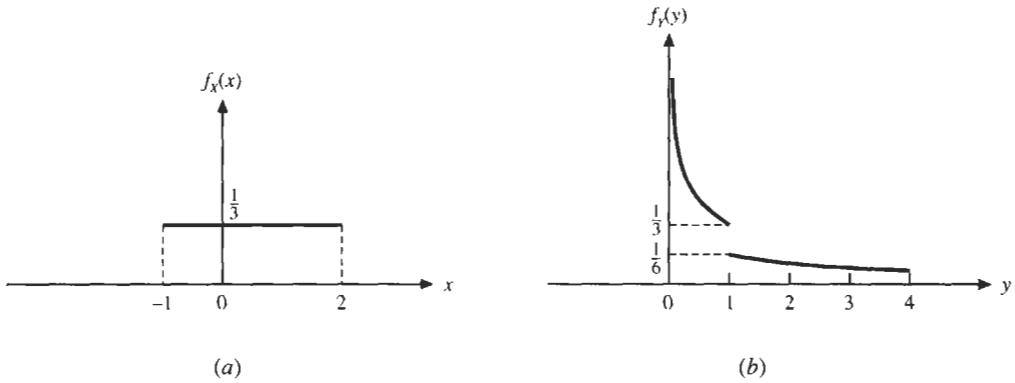
When $1 < y < 4$, $\sqrt{y}$ is in $R_X = (-1, 2)$ but $-\sqrt{y} < -1$, and by Eq. (*4.74*),

$$f_Y(y) = \frac{1}{2\sqrt{y}}\left(\frac{1}{3} + 0\right) = \frac{1}{6\sqrt{y}}$$

Thus,

$$f_Y(y) = \begin{cases} \dfrac{1}{3\sqrt{y}} & 0 < y < 1 \\ \dfrac{1}{6\sqrt{y}} & 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

which is sketched in Fig. 4-5(*b*).




(*a*) (*b*)

**Fig. 4-5**

**4.9.** Let $Y = e^X$. Find the pdf of $Y$ if $X$ is a uniform r.v. over (0, 1).

The pdf of $X$ is

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

The cdf of $Y$ is

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y)$$
$$= \int_{-\infty}^{\ln y} f_X(x)\,dx = \int_0^{\ln y} dx = \ln y \qquad 1 < y < e$$

Thus,
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \ln y = \frac{1}{y} \qquad 1 < y < e \tag{4.76}$$

*Alternative Solution:*

The function $y = g(x) = e^x$ is a continuous monotonically increasing function. Its inverse is $x = g^{-1}(y) = h(y) = \ln y$. Thus, by Eq. (*4.6*), we obtain

$$f_Y(y) = f_X(\ln y)\left|\frac{d}{dy} \ln y\right| = \frac{1}{y} f_X(\ln y) = \begin{cases} \dfrac{1}{y} & 0 < \ln y < 1 \\ 0 & \text{otherwise} \end{cases}$$

or
$$f_Y(y) = \begin{cases} \dfrac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

**4.10.** Let $Y = e^X$. Find the pdf of $Y$ if $X = N(\mu; \sigma^2)$.

The pdf of $X$ is [Eq. (*2.52*)]

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - \mu)^2\right] \qquad -\infty < x < \infty$$

Thus, using the technique shown in the alternative solution of Prob. 4.9, we obtain

$$f_Y(y) = \frac{1}{y} f_X(\ln y) = \frac{1}{y\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(\ln y - \mu)^2\right] \qquad 0 < y < \infty \tag{4.77}$$

Note that $X = \ln Y$ is the normal r.v.; hence, the r.v. $Y$ is called the *log-normal* r.v.

**4.11.** Let $Y = \tan X$. Find the pdf of $Y$ if $X$ is a uniform r.v. over $(-\pi/2, \pi/2)$.

The cdf of $X$ is [Eq. (*2.45*)]

$$F_X(x) = \begin{cases} 0 & x \le -\pi/2 \\ \dfrac{1}{\pi}(x + \pi/2) & -\pi/2 < x < \pi/2 \\ 1 & x \ge \pi/2 \end{cases}$$

Now
$$F_Y(y) = P(Y \le y) = P(\tan X \le y) = P(X \le \tan^{-1} y)$$
$$= F_X(\tan^{-1} y) = \frac{1}{\pi}\left(\tan^{-1} y + \frac{\pi}{2}\right) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1} y \qquad -\infty < y < \infty$$

Then the pdf of $Y$ is given by

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\pi(1 + y^2)} \qquad -\infty < y < \infty$$

Note that the r.v. $Y$ is a *Cauchy* r.v. with parameter 1.

**4.12.** Let $X$ be a continuous r.v. with the cdf $F_X(x)$. Let $Y = F_X(X)$. Show that $Y$ is a uniform r.v. over (0, 1).

Notice from the properties of a cdf that $y = F_X(x)$ is a monotonically nondecreasing function. Since $0 \le F_X(x) \le 1$ for all real $x$, $y$ takes on values only on the interval (0, 1). Using Eq. (*4.64*) (Prob. 4.2), we have

$$f_Y(y) = f_X(x)\frac{1}{dy/dx} = f_X(x)\frac{1}{dF_X(x)/dx} = \frac{f_X(x)}{f_X(x)} = 1 \qquad 0 < y < 1$$

Hence, $Y$ is a uniform r.v. over (0, 1).

**4.13.** Let $Y$ be a uniform r.v. over (0, 1). Let $F(x)$ be a function which has the properties of the cdf of a continuous r.v. with $F(a) = 0$, $F(b) = 1$, and $F(x)$ strictly increasing for $a < x < b$, where $a$ and $b$ could be $-\infty$ and $\infty$, respectively. Let $X = F^{-1}(Y)$. Show that the cdf of $X$ is $F(x)$.

$$F_X(x) = P(X \le x) = P[F^{-1}(Y) \le x]$$

Since $F(x)$ is strictly increasing, $F^{-1}(Y) \le x$ is equivalent to $Y \le F(x)$, and hence

$$F_X(x) = P(X \le x) = P[Y \le F(x)]$$

Now $Y$ is a uniform r.v. over (0, 1), and by Eq. (*2.45*),

$$F_Y(y) = P(Y \le y) = y \qquad 0 < y < 1$$

and accordingly,

$$F_X(x) = P(X \le x) = P[Y \le F(x)] = F(x) \qquad 0 < F(x) < 1$$

Note that this problem is the converse of Prob. 4.12.

**4.14.** Let $X$ be a continuous r.v. with the pdf

$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Find the transformation $Y = g(X)$ such that the pdf of $Y$ is

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The cdf of $X$ is

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi = \begin{cases} \displaystyle\int_0^x e^{-\xi}\, d\xi \\ 0 \end{cases} = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & x < 0 \end{cases}$$

Then from the result of Prob. 4.12, the r.v. $Z = 1 - e^{-X}$ is uniformly distributed over (0, 1). Similarly, the cdf of $Y$ is

$$F_Y(y) = \begin{cases} \displaystyle\int_0^y \frac{1}{2\sqrt{\eta}}\, d\eta \\ 0 \end{cases} = \begin{cases} \sqrt{y} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

and the r.v. $W = \sqrt{Y}$ is uniformly distributed over (0, 1). Thus, by setting $Z = W$, the required transformation is $Y = (1 - e^{-X})^2$.

## FUNCTIONS OF TWO RANDOM VARIABLES

**4.15.** Consider $Z = X + Y$. Show that if $X$ and $Y$ are independent Poisson r.v.'s with parameters $\lambda_1$ and $\lambda_2$, respectively, then $Z$ is also a Poisson r.v. with parameter $\lambda_1 + \lambda_2$.

We can write the event

$$(X + Y = n) = \bigcup_{i=0}^{n} (X = i,\ Y = n - i)$$

where events $(X = i,\ Y = n - i)$, $i = 0, 1, \ldots, n$, are disjoint. Since $X$ and $Y$ are independent, by Eqs. (*1.46*) and (*2.40*), we have

$$\begin{aligned} P(Z = n) = P(X + Y = n) &= \sum_{i=0}^{n} P(X = i,\ Y = n - i) = \sum_{i=0}^{n} P(X = i)P(Y = n - i) \\ &= \sum_{i=0}^{n} e^{-\lambda_1} \frac{\lambda_1^{\,i}}{i!}\, e^{-\lambda_2} \frac{\lambda_2^{n-i}}{(n-i)!} = e^{-(\lambda_1 + \lambda_2)} \sum_{i=0}^{n} \frac{\lambda_1^{\,i} \lambda_2^{n-i}}{i!\,(n-i)!} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} \sum_{i=0}^{n} \frac{n!}{i!\,(n-i)!}\, \lambda_1^{\,i} \lambda_2^{n-i} \\ &= \frac{e^{-(\lambda_1 + \lambda_2)}}{n!} (\lambda_1 + \lambda_2)^n \end{aligned}$$

which indicates that $Z = X + Y$ is a Poisson r.v. with $\lambda_1 + \lambda_2$.

**4.16.** Consider two r.v.'s $X$ and $Y$ with joint pdf $f_{XY}(x, y)$. Let $Z = X + Y$.

(*a*) Determine the pdf of $Z$.

(*b*) Determine the pdf of $Z$ if $X$ and $Y$ are independent.

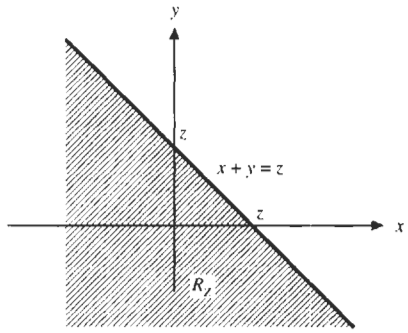



**Fig. 4-6**

(*a*) The range $R_Z$ of $Z$ corresponding to the event $(Z \le z) = (X + Y \le z)$ is the set of points $(x, y)$ which lie on and to the left of the line $z = x + y$ (Fig. 4-6). Thus, we have

$$F_Z(z) = P(X + Y \le z) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{z-x} f_{XY}(x, y)\, dy \right] dx \tag{4.78}$$

Then

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{-\infty}^{\infty} \left[ \frac{d}{dz} \int_{-\infty}^{z-x} f_{XY}(x, y)\, dy \right] dx$$

$$= \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx \tag{4.79}$$

(*b*) If $X$ and $Y$ are independent, then Eq. (*4.79*) reduces to

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx \tag{4.80a}$$

The integral on the right-hand side of Eq. (*4.80a*) is known as a *convolution* of $f_X(z)$ and $f_Y(z)$. Since the convolution is commutative, Eq. (*4.80a*) can also be written as

$$f_Z(z) = \int_{-\infty}^{\infty} f_Y(y) f_X(z - y)\, dy \tag{4.80b}$$

**4.17.** Using Eqs. (*4.19*) and (*3.30*), redo Prob. 4.16(*a*); that is, find the pdf of $Z = X + Y$.

Let $Z = X + Y$ and $W = X$. The transformation $z = x + y$, $w = x$ has the inverse transformation $x = w$, $y = z - w$, and

$$J(x, y) = \begin{vmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

By Eq. (*4.19*), we obtain

$$f_{ZW}(z, w) = f_{XY}(w, z - w)$$

Hence, by Eq. (*3.30*), we get

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z, w)\, dw = \int_{-\infty}^{\infty} f_{XY}(w, z - w)\, dw = \int_{-\infty}^{\infty} f_{XY}(x, z - x)\, dx$$

**4.18.** Suppose that $X$ and $Y$ are independent standard normal r.v.'s. Find the pdf of $Z = X + Y$.

The pdf's of $X$ and $Y$ are

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \qquad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

Then, by Eq. (*4.80a*), we have

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-(z-x)^2/2}\, dx$$
$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(z^2 - 2zx + 2x^2)/2}\, dx$$

Now, $z^2 - 2zx + 2x^2 = (\sqrt{2}\,x - z/\sqrt{2})^2 + z^2/2$, and we have

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\sqrt{2}\,x - z/\sqrt{2})^2/2}\, dx$$
$$= \frac{1}{\sqrt{2\pi}} e^{-z^2/4} \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\, du$$

with the change of variables $u = \sqrt{2}\,x - z/\sqrt{2}$. Since the integrand is the pdf of $N(0; 1)$, the integral is equal to unity, and we get

$$f_Z(z) = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/4} = \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/2(\sqrt{2})^2}$$

which is the pdf of $N(0; 2)$. Thus, $Z$ is a normal r.v. with zero mean and variance 2.

**4.19.** Let $X$ and $Y$ be independent uniform r.v.'s over (0, 1). Find and sketch the pdf of $Z = X + Y$.

Since $X$ and $Y$ are independent, we have

$$f_{XY}(x, y) = f_X(x) f_Y(y) = \begin{cases} 1 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of $Z$ is (0, 2), and

$$F_Z(z) = P(X + Y \le z) = \iint_{x+y \le z} f_{XY}(x, y)\, dx\, dy = \iint_{x+y \le z} dx\, dy$$

If $0 < z < 1$ [Fig. 4-7(*a*)],

$$F_Z(z) = \iint_{x+y<z} dx\, dy = \text{shaded area} = \frac{z^2}{2}$$

and

$$f_Z(z) = \frac{d}{dz} F_Z(z) = z$$

If $1 < z < 2$ [Fig. 4-7(*b*)],

$$F_Z(z) = \iint_{x+y<z} dx\, dy = \text{shaded area} = 1 - \frac{(2 - z)^2}{2}$$

and

$$f_Z(z) = \frac{d}{dz} F_Z(z) = 2 - z$$

Hence,

$$f_Z(z) = \begin{cases} z & 0 < z < 1 \\ 2 - z & 1 < z < 2 \\ 0 & \text{otherwise} \end{cases}$$

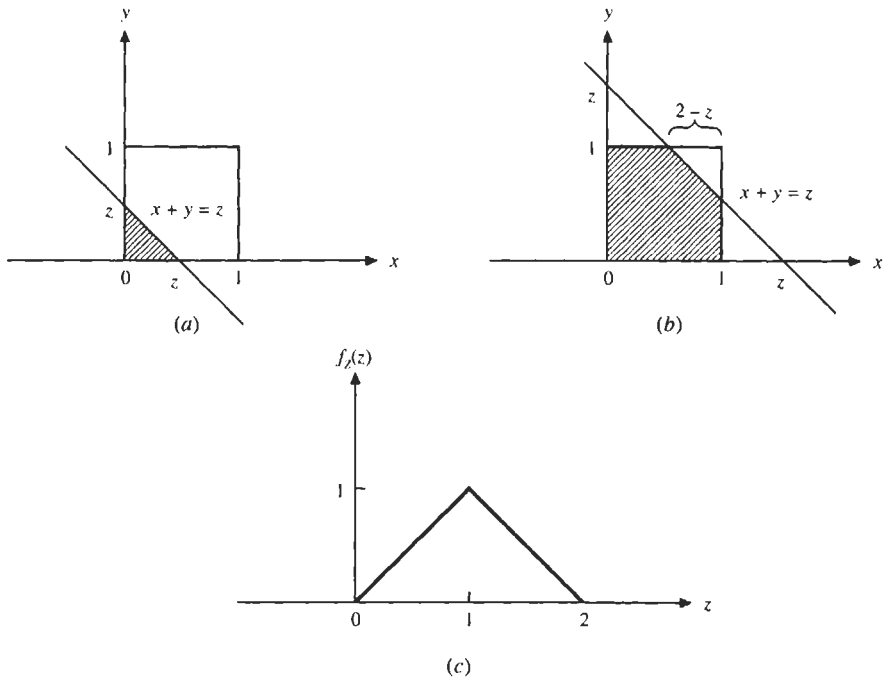



**Fig. 4-7**

which is sketched in Fig. 4-7(*c*). Note that the same result can be obtained by the convolution of $f_X(z)$ and $f_Y(z)$.

**4.20.** Let $X$ and $Y$ be independent gamma r.v.'s with respective parameters $(\alpha, \lambda)$ and $(\beta, \lambda)$. Show that $Z = X + Y$ is also a gamma r.v. with parameters $(\alpha + \beta, \lambda)$.

From Eq. (2.76) (Prob. 2.24),

$$f_X(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x > 0 \\ 0 & x < 0 \end{cases}$$

$$f_Y(y) = \begin{cases} \dfrac{\lambda e^{-\lambda y}(\lambda y)^{\beta-1}}{\Gamma(\beta)} & y > 0 \\ 0 & y < 0 \end{cases}$$

The range of $Z$ is $(0, \infty)$, and using Eq. (*4.80a*), we have

$$f_Z(z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^z \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} \lambda e^{-\lambda(z-x)}[\lambda(z-x)]^{\beta-1}\, dx$$

$$= \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda z} \int_0^z x^{\alpha-1}(z-x)^{\beta-1}\, dx$$

By the change of variable $w = x/z$, we have

$$f_Z(z) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} e^{-\lambda z} z^{\alpha+\beta-1} \int_0^1 w^{\alpha-1}(1-w)^{\beta-1}\, dw$$

$$= k e^{-\lambda z} z^{\alpha+\beta-1}$$

where $k$ is a constant which does not depend on $z$. The value of $k$ is determined as follows: Using Eq. (*2.22*) and definition (*2.77*) of the gamma function, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f_Z(z)\,dz &= k\int_0^{\infty} e^{-\lambda z}z^{\alpha+\beta-1}\,dz \\ &= \frac{k}{\lambda^{\alpha+\beta}}\int_0^{\infty} e^{-v}v^{\alpha+\beta-1}\,dv \qquad (\lambda z = v) \\ &= \frac{k}{\lambda^{\alpha+\beta}}\,\Gamma(\alpha+\beta) = 1\end{aligned}$$

Hence, $k = \lambda^{\alpha+\beta}/\Gamma(\alpha+\beta)$ and

$$f_Z(z) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha+\beta)}\,e^{-\lambda z}z^{\alpha+\beta-1} = \frac{\lambda e^{-\lambda z}(\lambda z)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \qquad z > 0$$

which indicates that $Z$ is a gamma r.v. with parameters $(\alpha+\beta, \lambda)$.

**4.21.** Consider two r.v.'s $X$ and $Y$ with joint pdf $f_{XY}(x, y)$. Determine the pdf of $Z = XY$.

Let $Z = XY$ and $W = X$. The transformation $z = xy$, $w = x$ has the inverse transformation $x = w$, $y = z/w$, and

$$J(z, w) = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ \dfrac{1}{w} & -\dfrac{z}{w^2} \end{vmatrix} = -\frac{1}{w}$$

Thus, by Eq. (*4.23*), we obtain

$$f_{ZW}(z, w) = \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right) \tag{4.81}$$

and the marginal pdf of $Z$ is

$$f_Z(z) = \int_{-\infty}^{\infty} \left|\frac{1}{w}\right| f_{XY}\left(w, \frac{z}{w}\right) dw \tag{4.82}$$

**4.22.** Let $X$ and $Y$ be independent uniform r.v.'s over (0, 1). Find the pdf of $Z = XY$.

We have

$$f_{XY}(x, y) = \begin{cases} 1 & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

The range of $Z$ is (0, 1). Then

$$f_{XY}\left(w, \frac{z}{w}\right) = \begin{cases} 1 & 0 < w < 1,\ 0 < z/w < 1 \\ 0 & \text{otherwise} \end{cases}$$

or

$$f_{XY}\left(w, \frac{z}{w}\right) = \begin{cases} 1 & 0 < z < w < 1 \\ 0 & \text{otherwise} \end{cases}$$

By Eq. (*4.82*),

$$f_Z(z) = \int_z^1 \frac{1}{w}\,dw = -\ln z \qquad 0 < z < 1$$

Thus,
$$f_Z(z) = \begin{cases} -\ln z & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases}$$

**4.23.** Consider two r.v.'s $X$ and $Y$ with joint pdf $f_{XY}(x, y)$. Determine the pdf of $Z = X/Y$.

Let $Z = X/Y$ and $W = Y$. The transformation $z = x/y$, $w = y$ has the inverse transformation $x = zw$, $y = w$, and

$$J(z, w) = \begin{vmatrix} \dfrac{\partial x}{\partial z} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial z} & \dfrac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$$

Thus, by Eq. (*4.23*), we obtain

$$f_{ZW}(z, w) = |w| f_{XY}(zw, w) \tag{4.83}$$

and the marginal pdf of $Z$ is

$$f_Z(z) = \int_{-\infty}^{\infty} |w| f_{XY}(zw, w)\, dw \tag{4.84}$$

**4.24.** Let $X$ and $Y$ be independent standard normal r.v.'s. Find the pdf of $Z = X/Y$.

Since $X$ and $Y$ are independent, using Eq. (*4.84*), we have

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} |w| f_X(zw) f_Y(w)\, dw = \int_{-\infty}^{\infty} |w| \frac{1}{2\pi} e^{-w^2(1+z^2)/2}\, dw \\ &= \frac{1}{2\pi} \int_0^{\infty} w e^{-w^2(1+z^2)/2}\, dw - \frac{1}{2\pi} \int_{-\infty}^{0} w e^{-w^2(1+z^2)/2}\, dw \\ &= \frac{1}{\pi(1+z^2)} \qquad -\infty < z < \infty \end{aligned}$$

which is the pdf of a Cauchy r.v. with parameter 1.

**4.25.** Let $X$ and $Y$ be two r.v.'s with joint pdf $f_{XY}(x, y)$ and joint cdf $F_{XY}(x, y)$. Let $Z = \max(X, Y)$.

(*a*) Find the cdf of $Z$.

(*b*) Find the pdf of $Z$ if $X$ and $Y$ are independent.

(*a*) The region in the $xy$ plane corresponding to the event $\{\max(X, Y) \le z\}$ is shown as the shaded area in Fig. 4-8. Then

$$F_Z(z) = P(Z \le z) = P(X \le z, Y \le z) = F_{XY}(z, z) \tag{4.85}$$

(*b*) If $X$ and $Y$ are independent, then

$$F_Z(z) = F_X(z) F_Y(z)$$

and differentiating with respect to $z$ gives

$$f_Z(z) = f_X(z) F_Y(z) + F_X(z) f_Y(z) \tag{4.86}$$

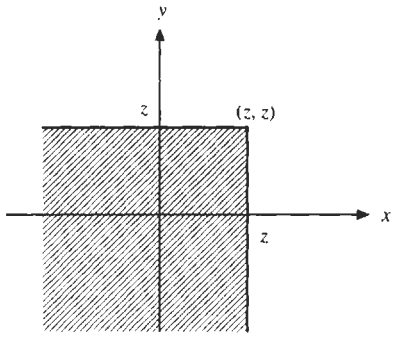



Fig. 4-8

**4.26.** Let $X$ and $Y$ be two r.v.'s with joint pdf $f_{XY}(x, y)$ and joint cdf $F_{XY}(x, y)$. Let $W = \min(X, Y)$.

(*a*) Find the cdf of $W$.

(*b*) Find the pdf of $W$ if $X$ and $Y$ are independent.

(*a*) The region in the $xy$ plane corresponding to the event $\{\min(X, Y) \leq w\}$ is shown as the shaded area in Fig. 4-9. Then

$$\begin{aligned} P(W \leq w) &= P\{(X \leq w) \cup (Y \leq w)\} \\ &= P(X \leq w) + P(Y \leq w) - P\{(X \leq w) \cap (Y \leq w)\} \end{aligned}$$

Thus,
$$F_W(w) = F_X(w) + F_Y(w) - F_{XY}(w, w) \tag{4.87}$$

(*b*) If $X$ and $Y$ are independent, then

$$F_W(w) = F_X(w) + F_Y(w) - F_X(w)F_Y(w)$$

and differentiating with respect to $w$ gives

$$\begin{aligned} f_W(w) &= f_X(w) + f_Y(w) - f_X(w)F_Y(w) - F_X(w)f_Y(w) \\ &= f_X(w)[1 - F_Y(w)] + f_Y(w)[1 - F_X(w)] \end{aligned} \tag{4.88}$$

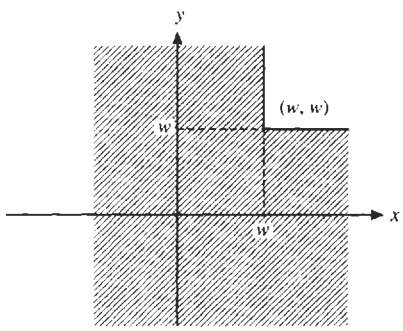



Fig. 4-9

**4.27.** Let $X$ and $Y$ be two r.v.'s with joint pdf $f_{XY}(x, y)$. Let

$$R = \sqrt{X^2 + Y^2} \qquad \Theta = \tan^{-1}\frac{Y}{X} \tag{4.89}$$

Find $f_{R\Theta}(r, \theta)$ in terms of $f_{XY}(x, y)$.

We assume that $r \geq 0$ and $0 \leq \theta < 2\pi$. With this assumption, the transformation

$$\sqrt{x^2 + y^2} = r \qquad \tan^{-1} \frac{y}{x} = \theta$$

has the inverse transformation

$$x = r \cos \theta \qquad y = r \sin \theta$$

Since

$$J(x, y) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

by Eq. (*4.23*) we obtain

$$f_{R\Theta}(r, \theta) = r f_{XY}(r \cos \theta, r \sin \theta) \tag{4.90}$$

**4.28.** A voltage $V$ is a function of time $t$ and is given by

$$V(t) = X \cos \omega t + Y \sin \omega t \tag{4.91}$$

in which $\omega$ is a constant angular frequency and $X = Y = N(0; \sigma^2)$ and they are independent.

(*a*) Show that $V(t)$ may be written as

$$V(t) = R \cos (\omega t - \Theta) \tag{4.92}$$

(*b*) Find the pdf's of r.v.'s $R$ and $\Theta$ and show that $R$ and $\Theta$ are independent.

(*a*) We have

$$\begin{aligned} V(t) &= X \cos \omega t + Y \sin \omega t \\ &= \sqrt{X^2 + Y^2} \left( \frac{X}{\sqrt{X^2 + Y^2}} \cos \omega t + \frac{Y}{\sqrt{X^2 + Y^2}} \sin \omega t \right) \\ &= \sqrt{X^2 + Y^2} (\cos \Theta \cos \omega t + \sin \Theta \sin \omega t) \\ &= R \cos(\omega t - \Theta) \end{aligned}$$

where $R = \sqrt{X^2 + Y^2}$ and $\Theta = \tan^{-1} \dfrac{Y}{X}$

which is the transformation (*4.89*).

(*b*) Since $X = Y = N(0; \sigma^2)$ and they are independent, we have

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2 + y^2)/(2\sigma^2)}$$

Thus, using Eq. (*4.90*), we get

$$f_{R\Theta}(r, \theta) = r f_{XY}(r \cos \theta, r \sin \theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/(2\sigma^2)} \tag{4.93}$$

Now

$$f_R(r) = \int_0^{2\pi} f_{R\Theta}(r, \theta)\, d\theta = \frac{r}{2\pi\sigma^2} e^{-r^2/(2\sigma^2)} \int_0^{2\pi} d\theta = \frac{r}{\sigma^2} e^{-r^2/(2\sigma^2)} \tag{4.94}$$

$$f_\Theta(\theta) = \int_0^{\infty} f_{R\Theta}(r, \theta)\, dr = \frac{1}{2\pi\sigma^2} \int_0^{\infty} r e^{-r^2/(2\sigma^2)}\, dr = \frac{1}{2\pi} \tag{4.95}$$

and $f_{R\Theta}(r, \theta) = f_R(r) f_\Theta(\theta)$; hence, $R$ and $\Theta$ are independent.

Note that $R$ is a *Rayleigh* r.v. (Prob. 2.23), and $\Theta$ is a uniform r.v. over $(0, 2\pi)$.

## FUNCTIONS OF $N$ RANDOM VARIABLES

**4.29.** Let $X$, $Y$, and $Z$ be independent standard normal r.v.'s. Let $W = (X^2 + Y^2 + Z^2)^{1/2}$. Find the pdf of $W$.

We have

$$f_{XYZ}(x, y, z) = f_X(x) f_Y(y) f_Z(z) = \frac{1}{(2\pi)^{3/2}} e^{-(x^2+y^2+z^2)/2}$$

and

$$F_W(w) = P(W \le w) = P(X^2 + Y^2 + Z^2 \le w^2) = \iiint_{R_W} \frac{1}{(2\pi)^{3/2}} e^{-(x^2+y^2+z^2)/2}\, dx\, dy\, dz$$

where $R_W = \{(x, y, z): x^2 + y^2 + z^2 \le w^2\}$. Using spherical coordinates (Fig. 4-10), we have

$$x^2 + y^2 + z^2 = r^2$$

$$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

and

$$F_W(w) = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \int_0^{\pi} \int_0^{w} e^{-r^2/2} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{w} e^{-r^2/2} r^2\, dr$$

$$= \frac{1}{(2\pi)^{3/2}} (2\pi)(2) \int_0^{w} r^2 e^{-r^2/2}\, dr \tag{4.96}$$

Thus, the pdf of $W$ is

$$f_W(w) = \frac{d}{dw} F_W(w) = \begin{cases} \sqrt{\dfrac{2}{\pi}}\, w^2 e^{-w^2/2} & w > 0 \\ 0 & w < 0 \end{cases} \tag{4.97}$$

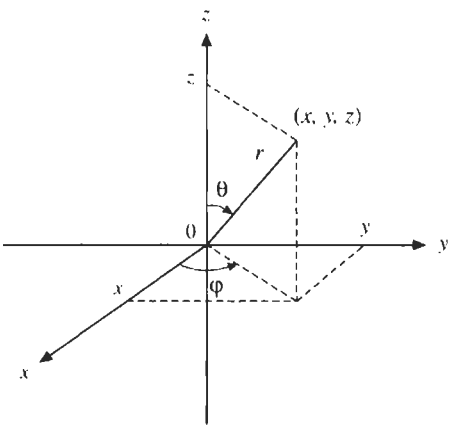



**Fig. 4-10** Spherical coordinates.

**4.30.** Let $X_1, \ldots, X_n$ be $n$ independent r.v.'s each with the identical pdf $f(x)$. Let $Z = \max(X_1, \ldots, X_n)$. Find the pdf of $Z$.

The probability $P(z < Z < z + dz)$ is equal to the probability that one of the r.v.'s falls in $(z, z + dz)$ and all others are less than $z$. The probability that one of $X_i$ $(i = 1, \ldots, n)$ falls in $(z, z + dz)$ and all others

are all less than $z$ is

$$f(z)\,dz\left(\int_{-\infty}^{z} f(x)\,dx\right)^{n-1}$$

Since there are $n$ ways of choosing the variables to be maximum, we have

$$f_Z(z) = nf(z)\left(\int_{-\infty}^{z} f(x)\,dx\right)^{n-1} = nf(z)[F(z)]^{n-1} \tag{4.98}$$

When $n = 2$, Eq. (4.98) reduces to

$$f_Z(z) = 2f(z)\int_{-\infty}^{z} f(x)\,dx = 2f(z)F(z) \tag{4.99}$$

which is the same as Eq. (4.86) (Prob. 4.25) with $f_X(z) = f_Y(z) = f(z)$ and $F_X(z) = F_Y(z) = F(z)$.

**4.31.** Let $X_1, \ldots, X_n$ be $n$ independent r.v.'s each with the identical pdf $f(x)$. Let $W = \min(X_1, \ldots, X_n)$. Find the pdf of $W$.

The probability $P(w < W < w + dw)$ is equal to the probability that one of the r.v.'s falls in $(w, w + dw)$ and all others are greater than $w$. The probability that one of $X_i$ $(i = 1, \ldots, n)$ falls in $(w, w + dw)$ and all others are greater than $w$ is

$$f(w)\,dw\left(\int_{w}^{\infty} f(x)\,dx\right)^{n-1}$$

Since there are $n$ ways of choosing the variables to be minimum, we have

$$f_W(w) = nf(w)\left(\int_{w}^{\infty} f(x)\,dx\right)^{n-1} = nf(z)[1 - F(z)]^{n-1} \tag{4.100}$$

When $n = 2$, Eq. (4.100) reduces to

$$f_W(w) = 2f(w)\int_{w}^{\infty} f(x)\,dx = 2f(w)[1 - F(w)] \tag{4.101}$$

which is the same as Eq. (4.88) (Prob. 4.26) with $f_X(w) = f_Y(w) = f(w)$ and $F_X(w) = F_Y(w) = F(w)$.

**4.32.** Let $X_i$, $i = 1, \ldots, n$, be $n$ independent gamma r.v.'s with respective parameters $(\alpha_i, \lambda)$, $i = 1, \ldots, n$. Let

$$Y = X_1 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

Show that $Y$ is also a gamma r.v. with parameters $(\sum_{i=1}^{n} \alpha_i, \lambda)$.

We prove this proposition by induction. Let us assume that the proposition is true for $n = k$; that is,

$$Z = X_1 + \cdots + X_k = \sum_{i=1}^{k} X_i$$

is a gamma r.v. with parameters $(\beta, \lambda) = (\sum_{i=1}^{k} \alpha_i, \lambda)$.

Let

$$W = Z + X_{k+1} = \sum_{i=1}^{k+1} X_i$$

Then, by the result of Prob. 4.20, we see that $W$ is a gamma r.v. with parameters $(\beta + \alpha_{k+1}, \lambda) = (\sum_{i=1}^{k+1} \alpha_i, \lambda)$. Hence, the proposition is true for $n = k + 1$. Next, by the result of Prob. 4.20, the proposition is true for $n = 2$. Thus, we conclude that the proposition is true for any $n \geq 2$.

**4.33.** Let $X_1, \ldots, X_n$ be $n$ independent exponential r.v.'s each with parameter $\lambda$. Let

$$Y = X_1 + \cdots + X_n = \sum_{i=1}^{n} X_i$$

Show that $Y$ is a gamma r.v. with parameters $(n, \lambda)$.

We note that an exponential r.v. with parameter $\lambda$ is a gamma r.v. with parameters $(1, \lambda)$ (Prob. 2.24). Thus, from the result of Prob. 4.32 and setting $\alpha_i = 1$, we conclude that $Y$ is a gamma r.v. with parameters $(n, \lambda)$.

**4.34.** Let $Z_1, \ldots, Z_n$ be $n$ independent standard normal r.v.'s. Let

$$Y = Z_1^2 + \cdots + Z_n^2 = \sum_{i=1}^{n} Z_i^2$$

Find the pdf of $Y$.

Let $Y_i = Z_i^2$. Then by Eq. (*4.75*) (Prob. 4.7), the pdf of $Y_i$ is

$$f_{Y_i}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}} e^{-y/2} & y > 0 \\ 0 & y < 0 \end{cases}$$

Now, using Eq. (*2.80*), we can rewrite

$$\frac{1}{\sqrt{2\pi y}} e^{-y/2} = \frac{\frac{1}{2}e^{-y/2}(y/2)^{1/2-1}}{\sqrt{\pi}} = \frac{\frac{1}{2}e^{-y/2}(y/2)^{1/2-1}}{\Gamma(\frac{1}{2})}$$

and we recognize the above as the pdf of a gamma r.v. with parameters $(\frac{1}{2}, \frac{1}{2})$ [Eq. (*2.76*)]. Thus, by the result of Prob. 4.32, we conclude that $Y$ is the gamma r.v. with parameters $(n/2, \frac{1}{2})$ and

$$f_Y(y) = \begin{cases} \dfrac{\frac{1}{2}e^{-y/2}(y/2)^{n/2-1}}{\Gamma(n/2)} = \dfrac{e^{-y/2}y^{n/2-1}}{2^{n/2}\Gamma(n/2)} & y > 0 \\ 0 & y < 0 \end{cases} \tag{4.102}$$

When $n$ is an even integer, $\Gamma(n/2) = [(n/2) - 1]!$, whereas when $n$ is odd, $\Gamma(n/2)$ can be obtained from $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$ [Eq. (*2.78*)] and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ [Eq. (*2.80*)].

Note that Equation (*4.102*) is referred to as the *chi-square* ($\chi^2$) density function with $n$ degrees of freedom, and $Y$ is known as the *chi-square* ($\chi^2$) r.v. with $n$ degrees of freedom. It is important to recognize that the sum of the squares of $n$ independent standard normal r.v.'s is a chi-square r.v. with $n$ degrees of freedom. The chi-square distribution plays an important role in statistical analysis.

**4.35.** Let $X_1$, $X_2$, and $X_3$ be independent standard normal r.v.'s. Let

$$\begin{aligned} Y_1 &= X_1 + X_2 + X_3 \\ Y_2 &= X_1 - X_2 \\ Y_3 &= X_2 - X_3 \end{aligned}$$

Determine the joint pdf of $Y_1$, $Y_2$, and $Y_3$.

Let

$$\begin{aligned} y_1 &= x_1 + x_2 + x_3 \\ y_2 &= x_1 - x_2 \\ y_3 &= x_2 - x_3 \end{aligned} \tag{4.103}$$

By Eq. (*4.32*), the jacobian of transformation (*4.103*) is

$$J(x_1, x_2, x_3) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = 3$$

Thus, solving the system (*4.103*), we get

$$x_1 = \tfrac{1}{3}(y_1 + 2y_2 + y_3)$$
$$x_2 = \tfrac{1}{3}(y_1 - y_2 + y_3)$$
$$x_3 = \tfrac{1}{3}(y_1 - y_2 - 2y_3)$$

Then by Eq. (*4.31*), we obtain

$$f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \tfrac{1}{3} f_{X_1X_2X_3}\left(\frac{y_1 + 2y_2 + y_3}{3}, \frac{y_1 - y_2 + y_3}{3}, \frac{y_1 - y_2 - 2y_3}{3}\right) \tag{4.104}$$

Since $X_1$, $X_2$, and $X_3$ are independent,

$$f_{X_1X_2X_3}(x_1, x_2, x_3) = \prod_{i=1}^{3} f_{X_i}(x_i) = \frac{1}{(2\pi)^{3/2}} e^{-(x_1^2 + x_2^2 + x_3^2)/2}$$

Hence,
$$f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = \frac{1}{3(2\pi)^{3/2}} e^{-q(y_1, y_2, y_3)/2}$$

where
$$q(y_1, y_2, y_3) = \left(\frac{y_1 + 2y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - y_2 + y_3}{3}\right)^2 + \left(\frac{y_1 - y_2 - 2y_3}{3}\right)^2$$
$$= \tfrac{1}{3}y_1^2 + \tfrac{2}{3}y_2^2 + \tfrac{2}{3}y_3^2 + \tfrac{2}{3}y_2 y_3$$

## EXPECTATION

**4.36.** Let $X$ be a uniform r.v. over (0, 1) and $Y = e^X$.

(*a*) Find $E(Y)$ by using $f_Y(y)$.

(*b*) Find $E(Y)$ by using $f_X(x)$.

(*a*) From Eq. (*4.76*) (Prob. 4.9),

$$f_Y(y) = \begin{cases} \dfrac{1}{y} & 1 < y < e \\ 0 & \text{otherwise} \end{cases}$$

Hence,
$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_1^e dy = e - 1$$

(*b*) The pdf of $X$ is

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then, by Eq. (*4.33*),

$$E(Y) = \int_{-\infty}^{\infty} e^x f_X(x)\, dx = \int_0^1 e^x\, dx = e - 1$$

**4.37.** Let $Y = aX + b$, where $a$ and $b$ are constants. Show that

(*a*)
$$E(Y) = E(aX + b) = aE(X) + b \tag{4.105}$$

(*b*)
$$\mathrm{Var}(Y) = \mathrm{Var}(aX + b) = a^2\, \mathrm{Var}(X) \tag{4.106}$$

We verify for the continuous case. The proof for the discrete case is similar.

(*a*) By Eq. (*4.33*),

$$E(Y) = E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f_X(x)\, dx$$
$$= a\int_{-\infty}^{\infty} x f_X(x)\, dx + b\int_{-\infty}^{\infty} f_X(x)\, dx = aE(X) + b$$

(*b*) Using Eq. (*4.105*), we have

$$\begin{aligned}\mathrm{Var}(Y) = \mathrm{Var}(aX + b) &= E\{(aX + b - [aE(X) + b])^2\} \\ &= E\{a^2[X - E(X)]^2\} = a^2 E\{[X - E(X)]^2\} = a^2\,\mathrm{Var}(X)\end{aligned}$$

**4.38.** Verify Eq. (*4.39*).

Using Eqs. (*3.58*) and (*3.38*), we have

$$\begin{aligned}E[E(Y \mid X)] &= \int_{-\infty}^{\infty} E(Y \mid x) f_X(x)\,dx = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y f_{Y|X}(y \mid x)\,dy\right] f_X(x)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \frac{f_{XY}(x, y)}{f_X(x)} f_X(x)\,dx\,dy = \int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\right] dy \\ &= \int_{-\infty}^{\infty} y f_Y(y)\,dy = E[Y]\end{aligned}$$

**4.39.** Let $Z = aX + bY$, where $a$ and $b$ are constants. Show that

$$E(Z) = E(aX + bY) = aE(X) + bE(Y) \tag{4.107}$$

We verify for the continuous case. The proof for the discrete case is similar.

$$\begin{aligned}E(Z) = E(aX + bY) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax + by) f_{XY}(x, y)\,dx\,dy \\ &= a\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x, y)\,dx\,dy + b\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x, y)\,dx\,dy \\ &= a\int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x, y)\,dy\right] dx + b\int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x, y)\,dx\right] dy \\ &= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} y f_Y(y)\,dy = aE(X) + bE(Y)\end{aligned}$$

Note that Eq. (*4.107*) (the linearity of $E$) can be easily extended to $n$ r.v.'s:

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) \tag{4.108}$$

**4.40.** Let $Y = aX + b$.

(*a*) Find the covariance of $X$ and $Y$.

(*b*) Find the correlation coefficient of $X$ and $Y$.

(*a*) By Eq. (*4.107*), we have

$$E(XY) = E[X(aX + b)] = aE(X^2) + bE(X)$$
$$E(Y) = E(aX + b) = aE(X) + b$$

Thus, the covariance of $X$ and $Y$ is [Eq. (*3.51*)]

$$\begin{aligned}\mathrm{Cov}(X, Y) = \sigma_{XY} &= E(XY) - E(X)E(Y) \\ &= aE(X^2) + bE(X) - E(X)[aE(X) + b] \\ &= a\{E(X^2) - [E(X)]^2\} = a\sigma_X{}^2\end{aligned} \tag{4.109}$$

(*b*) By Eq. (*4.106*), we have $\sigma_Y = |a|\sigma_X$. Thus, the correlation coefficient of $X$ and $Y$ is [Eq. (*3.53*)]

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{a\sigma_X{}^2}{\sigma_X |a| \sigma_X} = \frac{a}{|a|} = \begin{cases} 1 & a > 0 \\ -1 & a < 0 \end{cases} \tag{4.110}$$

**4.41.** Verify Eq. (*4.36*).

Since $X$ and $Y$ are independent, we have

$$\begin{aligned} E[g(X)h(Y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_{XY}(x, y)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)f_X(x)f_Y(y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} g(x)f_X(x)\,dx \int_{-\infty}^{\infty} h(y)f_Y(y)\,dy \\ &= E[g(X)]E[h(Y)] \end{aligned}$$

The proof for the discrete case is similar.

**4.42.** Let $X$ and $Y$ be defined by

$$X = \cos\Theta \qquad Y = \sin\Theta$$

where $\Theta$ is a random variable uniformly distributed over $(0, 2\pi)$.

(*a*) Show that $X$ and $Y$ are uncorrelated.

(*b*) Show that $X$ and $Y$ are not independent.

(*a*) We have

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & 0 < \theta < 2\pi \\ 0 & \text{otherwise} \end{cases}$$

Then
$$E(X) = \int_{-\infty}^{\infty} xf_X(x)\,dx = \int_0^{2\pi} \cos\theta\, f_\Theta(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\,d\theta = 0$$

Similarly,
$$E(Y) = \frac{1}{2\pi}\int_0^{2\pi} \sin\theta\,d\theta = 0$$

$$E(XY) = \frac{1}{2\pi}\int_0^{2\pi} \cos\theta\sin\theta\,d\theta = \frac{1}{4\pi}\int_0^{2\pi} \sin 2\theta\,d\theta = 0 = E(X)E(Y)$$

Thus, by Eq. (*3.52*), $X$ and $Y$ are uncorrelated.

(*b*)
$$E(X^2) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta\,d\theta = \frac{1}{4\pi}\int_0^{2\pi} (1 + \cos 2\theta)\,d\theta = \frac{1}{2}$$

$$E(Y^2) = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta\,d\theta = \frac{1}{4\pi}\int_0^{2\pi} (1 - \cos 2\theta)\,d\theta = \frac{1}{2}$$

$$E(X^2Y^2) = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\theta\sin^2\theta\,d\theta = \frac{1}{16\pi}\int_0^{2\pi} (1 - \cos 4\theta)\,d\theta = \frac{1}{8}$$

Hence

$$E(X^2Y^2) = \tfrac{1}{8} \neq \tfrac{1}{4} = E(X^2)E(Y^2)$$

If $X$ and $Y$ were independent, then by Eq. (*4.36*), we would have $E(X^2Y^2) = E(X^2)E(Y^2)$. Therefore, $X$ and $Y$ are not independent.

**4.43.** Let $X_1, \ldots, X_n$ be $n$ r.v.'s. Show that

$$\operatorname{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \operatorname{Cov}(X_i, X_j) \tag{4.111}$$

If $X_1, \ldots, X_n$ are pairwise independent, then

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \,\mathrm{Var}(X_i) \tag{4.112}$$

Let $$Y = \sum_{i=1}^{n} a_i X_i$$

Then by Eq. (*4.108*), we have

$$\begin{aligned}
\mathrm{Var}(Y) &= E\{[Y - E(Y)]^2\} = E\left\{\left(\sum_{i=1}^{n} a_i[X_i - E(X_i)]\right)^2\right\} \\
&= E\left\{\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j[X_i - E(X_i)][X_j - E(X_j)]\right\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j E\{[X_i - E(X_i)][X_j - E(X_j)]\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \,\mathrm{Cov}(X_i, X_j)
\end{aligned}$$

If $X_1, \ldots, X_n$ are pairwise independent, then (Prob. 3.22)

$$\mathrm{Cov}(X_i, X_j) = \begin{cases} \mathrm{Var}(X_i) & i = j \\ 0 & i \neq j \end{cases}$$

and Eq. (*4.111*) reduces to

$$\mathrm{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \,\mathrm{Var}(X_i)$$

## MOMENT GENERATING FUNCTIONS

**4.44.** Let the moment of a discrete r.v. $X$ be given by

$$E(X^k) = 0.8 \qquad k = 1, 2, \ldots$$

(*a*) Find the moment generating function of $X$.

(*b*) Find $P(X = 0)$ and $P(X = 1)$.

(*a*) By Eq. (*4.41*), the moment generating function of $X$ is

$$\begin{aligned}
M_X(t) &= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^k}{k!}E(X^k) + \cdots \\
&= 1 + 0.8\left(t + \frac{t^2}{2!} + \cdots + \frac{t^k}{k!} + \cdots\right) = 1 + 0.8\sum_{k=1}^{\infty}\frac{t^k}{k!} \\
&= 0.2 + 0.8\sum_{k=0}^{\infty}\frac{t^k}{k!} = 0.2 + 0.8e^t
\end{aligned} \tag{4.113}$$

(*b*) By definition (*4.40*),

$$M_X(t) = E(e^{tX}) = \sum_i e^{tx_i} p_X(x_i) \tag{4.114}$$

Thus, equating Eqs. (*4.113*) and (*4.114*), we obtain

$$p_X(0) = P(X = 0) = 0.2 \qquad p_X(1) = P(X = 1) = 0.8$$

**4.45.** Let $X$ be a Bernoulli r.v.

(*a*) Find the moment generating function of $X$.

(*b*) Find the mean and variance of $X$.

(*a*) By definition (*4.40*) and Eq. (*2.32*),

$$M_X(t) = E(e^{tX}) = \sum_i e^{tx_i} p_X(x_i)$$
$$= e^{t(0)} p_X(0) + e^{t(1)} p_X(1) = (1 - p) + pe^t \tag{4.115}$$

(*b*) By Eq. (*4.42*),

$$E(X) = M_X'(0) = pe^t \Big|_{t=0} = p$$

$$E(X^2) = M_X''(0) = pe^t \Big|_{t=0} = p$$

Hence, $$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = p - p^2 = p(1 - p)$$

**4.46.** Let $X$ be a binomial r.v. with parameters $(n, p)$.

(*a*) Find the moment generating function of $X$.

(*b*) Find the mean and variance of $X$.

(*a*) By definition (*4.40*) and Eq. (*2.36*), and letting $q = 1 - p$, we get

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k}$$
$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k q^{n-k} = (q + pe^t)^n \tag{4.116}$$

(*b*) The first two derivatives of $M_X(t)$ are

$$M_X'(t) = n(q + pe^t)^{n-1} pe^t$$
$$M_X''(t) = n(q + pe^t)^{n-1} pe^t + n(n - 1)(q + pe^t)^{n-2} (pe^t)^2$$

Thus, by Eq. (*4.42*),

$$\mu_X = E(X) = M_X'(0) = np$$
$$E(X^2) = M_X''(0) = np + n(n - 1)p^2$$

Hence, $$\sigma_X^2 = E(X^2) - [E(X)]^2 = np(1 - p)$$

**4.47.** Let $X$ be a Poisson r.v. with parameter $\lambda$.

(*a*) Find the moment generating function of $X$.

(*b*) Find the mean and variance of $X$.

(*a*) By definition (*4.40*) and Eq. (*2.40*),

$$M_X(t) = E(e^{tX}) = \sum_{i=0}^{\infty} e^{ti} e^{-\lambda} \frac{\lambda^i}{i!}$$
$$= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda e^t)^i}{i!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)} \tag{4.117}$$

(*b*) The first two derivatives of $M_X(t)$ are

$$M_X'(t) = \lambda e^t e^{\lambda(e^t - 1)}$$
$$M_X''(t) = (\lambda e^t)^2 e^{\lambda(e^t - 1)} + \lambda e^t e^{\lambda(e^t - 1)}$$

Thus, by Eq. (*4.42*),

$$E(X) = M_X'(0) = \lambda \qquad E(X^2) = M_X''(0) = \lambda^2 + \lambda$$

Hence, $$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

**4.48.** Let $X$ be an exponential r.v. with parameter $\lambda$.

(*a*) Find the moment generating function of $X$.

(*b*) Find the mean and variance of $X$.

(*a*) By definition (*4.40*) and Eq. (*2.48*),

$$M_X(t) = E(e^{tX}) = \int_0^\infty \lambda e^{-\lambda x} e^{tx}\, dx$$
$$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big|_0^\infty = \frac{\lambda}{\lambda - t} \qquad \lambda > t \tag{4.118}$$

(*b*) The first two derivatives of $M_X(t)$ are

$$M_X'(t) = \frac{\lambda}{(\lambda - t)^2} \qquad M_X''(t) = \frac{2\lambda}{(\lambda - t)^3}$$

Thus, by Eq. (*4.42*),

$$E(X) = M_X'(0) = \frac{1}{\lambda} \qquad E(X^2) = M_X''(0) = \frac{2}{\lambda^2}$$

Hence, $$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

**4.49.** Find the moment generating function of the standard normal r.v. $X = N(0; 1)$ and calculate the first three moments of $X$.

By definition (*4.40*) and Eq. (*2.52*),

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{tx}\, dx$$

Combining the exponents and completing the square, that is,

$$-\frac{x^2}{2} + tx = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

we obtain

$$M_X(t) = e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(x-t)^2/2}\, dx = e^{t^2/2} \tag{4.119}$$

since the integrand is the pdf of $N(t; 1)$.

Differentiating $M_X(t)$ with respect to $t$ three times, we have

$$M_X'(t) = te^{t^2/2} \qquad M_X''(t) = (t^2 + 1)e^{t^2/2} \qquad M_X^{(3)}(t) = (t^3 + 3t)e^{t^2/2}$$

Thus, by Eq. (*4.42*),

$$E(X) = M_X'(0) = 0 \qquad E(X^2) = M_X''(0) = 1 \qquad E(X^3) = M_X^{(3)}(0) = 0$$

**4.50.** Let $Y = aX + b$. Let $M_X(t)$ be the moment generating function of $X$. Show that the moment generating function of $Y$ is given by

$$M_Y(t) = e^{tb} M_X(at) \tag{4.120}$$

By Eqs. (*4.40*) and (*4.105*),

$$M_Y(t) = E(e^{tY}) = E[e^{t(aX+b)}]$$
$$= e^{tb} E(e^{atX}) = e^{tb} M_X(at)$$

**4.51.** Find the moment generating function of a normal r.v. $N(\mu; \sigma^2)$.

If $X$ is $N(0; 1)$, then from Prob. 4.1 (or Prob. 4.37), we see that $Y = \sigma X + \mu$ is $N(\mu; \sigma^2)$. Then by setting $a = \sigma$ and $b = \mu$ in Eq. (*4.120*) (Prob. 4.50) and using Eq. (*4.119*), we get

$$M_Y(t) = e^{\mu t} M_X(\sigma t) = e^{\mu t} e^{(\sigma t)^2/2} = e^{\mu t + \sigma^2 t^2/2} \tag{4.121}$$

**4.52.** Let $X_1, \ldots, X_n$ be $n$ independent r.v.'s and let the moment generating function of $X_i$ be $M_{X_i}(t)$. Let $Y = X_1 + \cdots + X_n$. Find the moment generating function of $Y$.

By definition (*4.40*),

$$\begin{aligned} M_Y(t) = E(e^{tY}) &= E[e^{t(X_1 + \cdots + X_n)}] = E(e^{tX_1} \cdots e^{tX_n}) \\ &= E(e^{tX_1}) \cdots E(e^{tX_n}) \qquad \text{(independence)} \\ &= M_{X_1}(t) \cdots M_{X_n}(t) \end{aligned} \tag{4.122}$$

**4.53.** Show that if $X_1, \ldots, X_n$ are independent Bernoulli r.v.'s with the parameter $p$, then $Y = X_1 + \cdots + X_n$ is a binomial r.v. with the parameters $(n, p)$.

Using Eqs. (*4.122*) and (*4.115*), the moment generating function of $Y$ is

$$M_Y(t) = \prod_{i=1}^{n} (q + pe^t) = (q + pe^t)^n \qquad q = 1 - p$$

which is the moment generating function of a binomial r.v. with parameters $(n, p)$ [Eq. (*4.116*)]. Hence, $Y$ is a binomial r.v. with parameters $(n, p)$.

**4.54.** Show that if $X_1, \ldots, X_n$ are independent Poisson r.v.'s $X_i$ having parameter $\lambda_i$, then $Y = X_1 + \cdots + X_n$ is also a Poisson r.v. with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$.

Using Eqs. (*4.122*) and (*4.117*), the moment generating function of $Y$ is

$$M_Y(t) = \prod_{i=1}^{n} e^{\lambda_i(e^t - 1)} = e^{(\Sigma\lambda_i)(e^t - 1)} = e^{\lambda(e^t - 1)}$$

which is the moment generating function of a Poisson r.v. with parameter $\lambda$. Hence, $Y$ is a Poisson r.v. with parameter $\lambda = \Sigma\lambda_i = \lambda_1 + \cdots + \lambda_n$.

Note that Prob. 4.15 is a special case for $n = 2$.

**4.55.** Show that if $X_1, \ldots, X_n$ are independent normal r.v.'s and $X_i = N(\mu_i; \sigma_i^2)$, then $Y = X_1 + \cdots + X_n$ is also a normal r.v. with mean $\mu = \mu_1 + \cdots + \mu_n$ and variance $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

Using Eqs. (*4.122*) and (*4.121*), the moment generating function of $Y$ is

$$M_Y(t) = \prod_{i=1}^{n} e^{(\mu_i t + \sigma_i^2 t^2/2)} = e^{(\Sigma\mu_i)t + (\Sigma\sigma_i^2)t^2/2} = e^{\mu t + \sigma^2 t^2/2}$$

which is the moment generating function of a normal r.v. with mean $\mu$ and variance $\sigma^2$. Hence, $Y$ is a normal r.v. with mean $\mu = \mu_1 + \cdots + \mu_n$ and variance $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

Note that Prob. 4.18 is a special case for $n = 2$ with $\mu_i = 0$ and $\sigma_i^2 = 1$.

**4.56.** Find the moment generating function of a gamma r.v. $Y$ with parameters $(n, \lambda)$.

From Prob. 4.33, we see that if $X_1, \ldots, X_n$ are independent exponential r.v.'s, each with parameter $\lambda$, then $Y = X_1 + \cdots + X_n$ is a gamma r.v. with parameters $(n, \lambda)$. Thus, by Eqs. (*4.122*) and (*4.118*), the moment generating function of $Y$ is

$$M_Y(t) = \prod_{i=1}^{n} \left(\frac{\lambda}{\lambda - t}\right) = \left(\frac{\lambda}{\lambda - t}\right)^n \tag{4.123}$$

## CHARACTERISTIC FUNCTIONS

**4.57.** The r.v. $X$ can take on the values $x_1 = -1$ and $x_2 = +1$ with pmf's $p_X(x_1) = p_X(x_2) = 0.5$. Determine the characteristic function of $X$.

By definition (*4.50*), the characteristic function of $X$ is

$$\Psi_X(\omega) = 0.5e^{-j\omega} + 0.5e^{j\omega} = \tfrac{1}{2}(e^{j\omega} + e^{-j\omega}) = \cos\omega$$

**4.58.** Find the characteristic function of a Cauchy r.v. $X$ with parameter $a$ and pdf given by

$$f_X(x) = \frac{a}{\pi(x^2 + a^2)} \qquad -\infty < x < \infty$$

By direct integration (or from the Table of Fourier transforms in Appendix B), we have the following Fourier transform pair:

$$e^{-a|x|} \leftrightarrow \frac{2a}{\omega^2 + a^2}$$

Now, by the duality property of the Fourier transform, we have the following Fourier transform pair:

$$\frac{2a}{x^2 + a^2} \leftrightarrow 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$$

or (by the linearity property of the Fourier transform)

$$\frac{a}{\pi(x^2 + a^2)} \leftrightarrow e^{-a|\omega|}$$

Thus, the characteristic function of $X$ is

$$\Psi_X(\omega) = e^{-a|\omega|} \tag{4.124}$$

Note that the moment generating function of the Cauchy r.v. $X$ does not exist, since $E(X^n) \to \infty$ for $n \geq 2$.

**4.59.** The characteristic function of a r.v. $X$ is given by

$$\Psi_X(\omega) = \begin{cases} 1 - |\omega| & |\omega| < 1 \\ 0 & |\omega| > 1 \end{cases}$$

Find the pdf of $X$.

From formula (*4.51*), we obtain the pdf of $X$ as

$$\begin{aligned} f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_X(\omega)e^{-j\omega x}\,d\omega \\ &= \frac{1}{2\pi}\left[\int_{-1}^{0}(1+\omega)e^{-j\omega x}\,d\omega + \int_{0}^{1}(1-\omega)e^{-j\omega x}\,d\omega\right] \\ &= \frac{1}{2\pi x^2}(2 - e^{jx} - e^{-jx}) = \frac{1}{\pi x^2}(1 - \cos x) \\ &= \frac{1}{2\pi}\left[\frac{\sin(x/2)}{x/2}\right]^2 \qquad -\infty < x < \infty \end{aligned}$$

**4.60.** Find the characteristic function of a normal r.v. $X = N(\mu; \sigma^2)$.

The moment generating function of $N(\mu; \sigma^2)$ is [Eq. (*4.121*)]

$$M_X(t) = e^{\mu t + \sigma^2 t^2/2}$$

Thus the characteristic function of $N(\mu; \sigma^2)$ is obtained by setting $t = j\omega$ in $M_X(t)$; that is,

$$\Psi_X(\omega) = e^{\mu t + \sigma^2 t^2/2}\Big|_{t=j\omega} = e^{j\omega\mu - \sigma^2\omega^2/2} \tag{4.125}$$

**4.61.** Let $Y = aX + b$. Show that if $\Psi_X(\omega)$ is the characteristic function of $X$, then the characteristic function of $Y$ is given by

$$\Psi_Y(\omega) = e^{j\omega b}\Psi_X(a\omega) \tag{4.126}$$

By definition (4.50),

$$\begin{aligned}\Psi_Y(\omega) = E(e^{j\omega Y}) &= E[e^{j\omega(aX+b)}] \\ &= e^{j\omega b}E(e^{ja\omega X}) = e^{j\omega b}\Psi_X(a\omega)\end{aligned}$$

**4.62.** Using the characteristic equation technique, redo part (*b*) of Prob. 4.16.

Let $Z = X + Y$, where $X$ and $Y$ are independent. Then

$$\begin{aligned}\Psi_Z(\omega) = E(e^{j\omega Z}) &= E(e^{j\omega(X+Y)}) = E(e^{j\omega X})E(e^{j\omega Y}) \\ &= \Psi_X(\omega)\Psi_Y(\omega)\end{aligned} \tag{4.127}$$

Applying the convolution theorem of the Fourier transform (Appendix B), we obtain

$$\begin{aligned}f_Z(z) &= \mathcal{F}^{-1}[\Psi_Z(\omega)] = \mathcal{F}^{-1}[\Psi_X(\omega)\Psi_Y(\omega)] \\ &= f_X(z) * f_Y(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)\,dx\end{aligned}$$

## THE LAWS OF LARGE NUMBERS AND THE CENTRAL LIMIT THEOREM

**4.63.** Verify the weak law of large numbers (4.58); that is,

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0 \qquad \text{for any } \varepsilon$$

where $\bar{X}_n = \dfrac{1}{n}(X_1 + \cdots + X_n)$ and $E(X_i) = \mu$, $\mathrm{Var}(X_i) = \sigma^2$.

Using Eqs. (4.108) and (4.112), we have

$$E(\bar{X}_n) = \mu \qquad \text{and} \qquad \mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n} \tag{4.128}$$

Then it follows from Chebyshev's inequality [Eq. (2.97)] (Prob. 2.36) that

$$P(|\bar{X}_n - \mu| > \varepsilon) \le \frac{\sigma^2}{n\varepsilon^2} \tag{4.129}$$

Since $\lim_{n\to\infty} \sigma^2/(n\varepsilon^2) = 0$, we get

$$\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0$$

**4.64.** Let $X$ be a r.v. with pdf $f_X(x)$ and let $X_1, \ldots, X_n$ be a set of independent r.v.'s each with pdf $f_X(x)$. Then the set of r.v.'s $X_1, \ldots, X_n$ is called a *random sample of size n* of $X$. The *sample mean* is defined by

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{4.130}$$

Let $X_1, \ldots, X_n$ be a random sample of $X$ with mean $\mu$ and variance $\sigma^2$. How many samples of $X$ should be taken if the probability that the sample mean will not deviate from the true mean $\mu$ by more than $\sigma/10$ is at least 0.95?

Setting $\varepsilon = \sigma/10$ in Eq. (*4.129*), we have

$$P\left(|\bar{X}_n - \mu| > \frac{\sigma}{10}\right) = 1 - P\left(|\bar{X}_n - \mu| \le \frac{\sigma}{10}\right) \le \frac{\sigma^2}{n\sigma^2/100} = \frac{100}{n}$$

or

$$P\left(|\bar{X}_n - \mu| \le \frac{\sigma}{10}\right) \ge 1 - \frac{100}{n}$$

Thus if we want this probability to be at least 0.95, we must have $100/n \le 0.05$ or $n \ge 100/0.05 = 2000$.

**4.65.** Verify the central limit theorem (*4.61*).

Let $X_1, \ldots, X_n$ be a sequence of independent, identically distributed r.v.'s with $E(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$. Consider the sum $S_n = X_1 + \cdots + X_n$. Then by Eqs. (*4.108*) and (*4.112*), we have $E(S_n) = n\mu$ and $\mathrm{Var}(S_n) = n\sigma^2$. Let

$$Z_n = \frac{S_n - n\mu}{\sqrt{n}\,\sigma} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right) \tag{4.131}$$

Then by Eqs. (*4.105*) and (*4.106*), we have $E(Z_n) = 0$ and $\mathrm{Var}(Z_n) = 1$. Let $M(t)$ be the moment generating function of the standardized r.v. $Y_i = (X_i - \mu)/\sigma$. Since $E(Y_i) = 0$ and $E(Y_i^2) = \mathrm{Var}(Y_i) = 1$, by Eq. (*4.42*), we have

$$M(0) = 1 \qquad M'(0) = E(Y_i) = 0 \qquad M''(0) = E(Y_i^2) = 1$$

Given that $M'(t)$ and $M''(t)$ are continuous functions of $t$, a Taylor (or Maclaurin) expansion of $M(t)$ about $t = 0$ can be expressed as

$$M(t) = M(0) + M'(0)t + M''(t_1)\frac{t^2}{2!} = 1 + M''(t_1)\frac{t^2}{2} \qquad 0 \le t_1 \le t$$

By adding and subtracting $t^2/2$, we have

$$M(t) = 1 + \tfrac{1}{2}t^2 + \tfrac{1}{2}[M''(t_1) - 1]t^2 \tag{4.132}$$

Now, by Eqs. (*4.120*) and (*4.122*), the moment generating function of $Z_n$ is

$$M_{Z_n}(t) = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n \tag{4.133}$$

Using Eq. (*4.132*), Eq. (*4.133*) can be written as

$$M_{Z_n}(t) = \left[1 + \frac{1}{2}\left(\frac{t}{\sqrt{n}}\right)^2 + \frac{1}{2}[M''(t_1) - 1]\left(\frac{t}{\sqrt{n}}\right)^2\right]^n$$

where now $t_1$ is between 0 and $t/\sqrt{n}$. Since $M''(t)$ is continuous at $t = 0$ and $t_1 \to 0$ as $n \to \infty$, we have

$$\lim_{n\to\infty} [M''(t_1) - 1] = M''(0) - 1 = 1 - 1 = 0$$

Thus, from elementary calculus, $\lim_{n\to\infty}(1 + x/n)^n = e^x$, and we obtain

$$\begin{aligned}\lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty}\left\{1 + \frac{t^2}{2n} + \frac{1}{2n}[M''(t_1) - 1]t^2\right\}^n \\ &= \lim_{n\to\infty}\left(1 + \frac{t^2/2}{n}\right)^n = e^{t^2/2}\end{aligned}$$

The right-hand side is the moment generating function of the standard normal r.v. $Z = N(0; 1)$ [Eq. (*4.119*)]. Hence, by Lemma 4.2 of the moment generating function,

$$\lim_{n\to\infty} Z_n = N(0; 1)$$

**4.66.** Let $X_1, \ldots, X_n$ be $n$ independent Cauchy r.v.'s with identical pdf shown in Prob. 4.58. Let

$$Y_n = \frac{1}{n}(X_1 + \cdots + X_n) = \frac{1}{n}\sum_{n=1}^{n} X_i$$

(*a*) Find the characteristic function of $Y_n$.

(*b*) Find the pdf of $Y_n$.

(*c*) Does the central limit theorem hold?

(*a*) From Eq. (*4.124*), the characteristic function of $X_i$ is

$$\Psi_{X_i}(\omega) = e^{-a|\omega|}$$

Let $Y = X_1 + \cdots + X_n$. Then the characteristic function of $Y$ is

$$\Psi_Y(\omega) = E(e^{j\omega Y}) = E[e^{j\omega(X_1 + \cdots + X_n)}] = \prod_{i=1}^{n} \Psi_{X_i}(\omega) = e^{-na|\omega|} \tag{4.134}$$

Now $Y_n = (1/n)Y$. Thus, by Eq. (*4.126*), the characteristic function of $Y_n$ is

$$\Psi_{Y_n}(\omega) = \Psi_Y\left(\frac{\omega}{n}\right) = e^{-na|\omega/n|} = e^{-a|\omega|} \tag{4.135}$$

(*b*) Equation (*4.135*) indicates that $Y_n$ is also a Cauchy r.v. with parameter $a$, and its pdf is the same as that of $X_i$.

(*c*) Since the characteristic function of $Y_n$ is independent of $n$ and so is its pdf, $Y_n$ does not tend to a normal r.v. as $n \to \infty$, and so the central limit theorem does not hold in this case.

**4.67.** Let $Y$ be a binomial r.v. with parameters $(n, p)$. Using the central limit theorem, derive the approximation formula

$$P(Y \le y) \approx \Phi\left(\frac{y - np}{\sqrt{np(1-p)}}\right) \tag{4.136}$$

where $\Phi(z)$ is the cdf of a standard normal r.v. [Eq. (*2.54*)].

We saw in Prob. 4.53 that if $X_1, \ldots, X_n$ are independent Bernoulli r.v.'s, each with parameter $p$, then $Y = X_1 + \cdots + X_n$ is a binomial r.v. with parameters $(n, p)$. Since $X_i$'s are independent, we can apply the central limit theorem to the r.v. $Z_n$ defined by

$$Z_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - E(X_i)}{\mathrm{Var}(X_i)}\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{X_i - p}{\sqrt{p(1-p)}}\right) \tag{4.137}$$

Thus, for large $n$, $Z_n$ is normally distributed and

$$P(Z_n \le x) \approx \Phi(x) \tag{4.138}$$

Substituting Eq. (*4.137*) into Eq. (*4.138*) gives

$$P\left[\frac{1}{\sqrt{np(1-p)}}\left(\sum_{i=1}^{n}(X_i - p)\right) \le x\right] = P[Y \le x\sqrt{np(1-p)} + np] \approx \Phi(x)$$

or

$$P(Y \le y) \approx \Phi\left(\frac{y - np}{\sqrt{np(1-p)}}\right)$$

Because we are approximating a discrete distribution by a continuous one, a slightly better approximation is given by

$$P(Y \le y) \approx \Phi\left(\frac{y + \frac{1}{2} - np}{\sqrt{np(1-p)}}\right) \tag{4.139}$$

Formula (*4.139*) is referred to as a *continuity correction* of Eq. (*4.136*).

**4.68.** Let $Y$ be a Poisson r.v. with parameter $\lambda$. Using the central limit theorem, derive approximation formula:

$$P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right) \tag{4.140}$$

We saw in Prob. 4.54 that if $X_1, \ldots, X_n$ are independent Poisson r.v.'s $X_i$ having parameter $\lambda_i$, then $Y = X_1 + \cdots + X_n$ is also a Poisson r.v. with parameter $\lambda = \lambda_1 + \cdots + \lambda_n$. Using this fact, we can view a Poisson r.v. $Y$ with parameter $\lambda$ as a sum of independent Poisson r.v.'s $X_i$, $i = 1, \ldots, n$, each with parameter $\lambda/n$; that is,

$$Y = X_1 + \cdots + X_n$$

$$E(X_i) = \frac{\lambda}{n} = \mathrm{Var}(X_i)$$

The central limit theorem then implies that the r.v. $Z$ defined by

$$Z = \frac{Y - E(Y)}{\sqrt{\mathrm{Var}(Y)}} = \frac{Y-\lambda}{\sqrt{\lambda}} \tag{4.141}$$

is approximately normal and

$$P(Z \le z) \approx \Phi(z) \tag{4.142}$$

Substituting Eq. (4.141) into Eq. (4.142) gives

$$P\left(\frac{Y-\lambda}{\sqrt{\lambda}} \le z\right) = P(Y \le \sqrt{\lambda}\, z + \lambda) \approx \Phi(z)$$

or

$$P(Y \le y) \approx \Phi\left(\frac{y-\lambda}{\sqrt{\lambda}}\right)$$

Again, using a continuity correction, a slightly better approximation is given by

$$P(Y \le y) \approx \Phi\left(\frac{y+\frac{1}{2}-\lambda}{\sqrt{\lambda}}\right) \tag{4.143}$$

## Supplementary Problems

**4.69.** Let $Y = 2X + 3$. Find the pdf of $Y$ if $X$ is a uniform r.v. over $(-1, 2)$.

*Ans.* $f_Y(y) = \begin{cases} \frac{1}{6} & 1 < y < 7 \\ 0 & \text{otherwise} \end{cases}$

**4.70.** Let $X$ be a r.v. with pdf $f_X(x)$. Let $Y = |X|$. Find the pdf of $Y$ in terms of $f_X(x)$.

*Ans.* $f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & y > 0 \\ 0 & y < 0 \end{cases}$

**4.71.** Let $Y = \sin X$, where $X$ is uniformly distributed over $(0, 2\pi)$. Find the pdf of $Y$.

*Ans.* $f_Y(y) = \begin{cases} \dfrac{1}{\pi\sqrt{1-y^2}} & -1 < y < 1 \\ 0 & \text{otherwise} \end{cases}$

**4.72.** Let $X$ and $Y$ be independent r.v.'s, each uniformly distributed over (0, 1). Let $Z = X + Y$, $W = X - Y$. Find the marginal pdf's of Z and W.

*Ans.* $f_Z(z) = \begin{cases} z & 0 < z < 1 \\ -z + 2 & 1 < z < 2 \\ 0 & \text{otherwise} \end{cases}$ $f_W(w) = \begin{cases} w + 1 & -1 < w < 0 \\ -w + 1 & 0 < w < 1 \\ 0 & \text{otherwise} \end{cases}$

**4.73.** Let $X$ and $Y$ be independent exponential r.v.'s with parameters $\alpha$ and $\beta$, respectively. Find the pdf of (*a*) $Z = X - Y$; (*b*) $Z = X/Y$; (*c*) $Z = \max(X, Y)$; (*d*) $Z = \min(X, Y)$.

*Ans.* (*a*) $f_Z(z) = \begin{cases} \dfrac{\alpha\beta}{\alpha + \beta} e^{-\alpha z} & z > 0 \\ \dfrac{\alpha\beta}{\alpha + \beta} e^{\beta z} & z < 0 \end{cases}$ (*b*) $f_Z(z) = \begin{cases} \dfrac{\alpha\beta}{(\alpha z + \beta)^2} & z > 0 \\ 0 & z < 0 \end{cases}$

(*c*) $f_Z(z) = \begin{cases} \alpha e^{-\alpha z}(1 - e^{-\beta z}) + \beta e^{-\beta z}(1 - e^{-\alpha z}) & z > 0 \\ 0 & z < 0 \end{cases}$

(*d*) $f_Z(z) = \begin{cases} (\alpha + \beta)e^{-(\alpha + \beta)z} & z > 0 \\ 0 & z < 0 \end{cases}$

**4.74.** Let $X$ denote the number of heads obtained when three independent tossings of a fair coin are made. Let $Y = X^2$. Find $E(Y)$.

*Ans.* 3

**4.75.** Let $X$ be a uniform r.v. over $(-1, 1)$. Let $Y = X^n$.

(*a*) Calculate the covariance of $X$ and $Y$.

(*b*) Calculate the correlation coefficient of $X$ and $Y$.

*Ans.* (*a*) $\text{Cov}(X, Y) = \begin{cases} \dfrac{1}{n + 2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$ (*b*) $\rho_{XY} = \begin{cases} \dfrac{\sqrt{3(2n + 1)}}{n + 2} & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$

**4.76.** Let the moment generating function of a discrete r.v. $X$ be given by

$$M_X(t) = 0.25e^t + 0.35e^{3t} + 0.40e^{5t}$$

Find $P(X = 3)$.

*Ans.* 0.35

**4.77.** Let $X$ be a geometric r.v. with parameter $p$.

(*a*) Determine the moment generating function of $X$.

(*b*) Find the mean of $X$ for $p = \frac{2}{3}$.

*Ans.* (*a*) $M_X(t) = \dfrac{pe^t}{1 - qe^t}$ $t < -\ln q$, $q = 1 - p$ (*b*) $E(X) = \frac{3}{2}$

**4.78.** Let $X$ be a uniform r.v. over $(a, b)$.

(*a*) Determine the moment generating function of $X$.

(*b*) Using the result of (*a*), find $E(X)$, $E(X^2)$, and $E(X^3)$.

*Ans.* (*a*) $M_X(t) = \dfrac{e^{tb} - e^{ta}}{t(b - a)}$

(*b*) $E(X) = \frac{1}{2}(b + a)$, $E(X^2) = \frac{1}{3}(b^2 + ab + a^2)$ $E(X^3) = \frac{1}{4}(b^3 + b^2a + ba^2 + a^3)$

**4.79.** Consider a r.v. $X$ with pdf

$$f_X(x) = \frac{1}{\sqrt{32\pi}} e^{-(x+7)^2/32} \qquad -\infty < x < \infty$$

Find the moment generating function of $X$.

*Ans.* $M_X(t) = e^{-7t+8t^2}$

**4.80.** Let $X = N(0; 1)$. Using the moment generating function of $X$, determine $E(X^n)$.

*Ans.* $E(X^n) = \begin{cases} 0 & n = 1, 3, 5, \ldots \\ 1 \cdot 3 \cdot \cdots \cdot (n-1) & n = 2, 4, 6, \ldots \end{cases}$

**4.81.** Let $X$ and $Y$ be independent binomial r.v.'s with parameters $(n, p)$ and $(m, p)$, respectively. Let $Z = X + Y$. What is the distribution of $Z$?

*Hint:* Use the moment generating functions.

*Ans.* $Z$ is a binomial r.v. with parameters $(n + m, p)$.

**4.82.** Let $(X, Y)$ be a continuous bivariate r.v. with joint pdf

$$f_{XY}(x, y) = \begin{cases} e^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) Find the joint moment function of $X$ and $Y$.

(*b*) Find the joint moments $m_{10}$, $m_{01}$, and $m_{11}$.

*Ans.* (*a*) $M_{XY}(t_1, t_2) = \dfrac{1}{(1-t_1)(1-t_2)}$ (*b*) $m_{10} = 1, m_{01} = 1, m_{11} = 1$

**4.83.** Let $(X, Y)$ be a bivariate normal r.v. defined by Eq. (*3.88*). Find the joint moment generating function of $X$ and $Y$.

*Ans.* $M_{XY}(t_1, t_2) = e^{t_1\mu_X + t_2\mu_Y + (t_1^2\sigma_X^2 + 2t_1t_2\sigma_X\sigma_Y\rho + t_2^2\sigma_Y^2)/2}$

**4.84.** Let $X_1, \ldots, X_n$ be $n$ independent r.v.'s and $X_i > 0$. Let

$$Y = X_1 \cdot \cdots \cdot X_n = \prod_{i=1}^{n} X_i$$

Show that for large $n$, the pdf of $Y$ is approximately log-normal.

*Hint:* Take the natural logarithm of $Y$ and use the central limit theorem and the result of Prob. 4.10.

**4.85.** Let $Y = (X - \lambda)/\sqrt{\lambda}$, where $X$ is a Poisson r.v. with parameter $\lambda$. Show that $Y \approx N(0; 1)$ when $\lambda$ is sufficiently large.

*Hint:* Find the moment generating function of $Y$ and let $\lambda \to \infty$.

**4.86.** Consider an experiment of tossing a fair coin 1000 times. Find the probability of obtaining more that 520 heads (*a*) by using formula (*4.136*), and (*b*) by formula (*4.139*).

*Ans.* (*a*) 0.1038 (*b*) 0.0974

**4.87.** The number of cars entering a parking lot is Poisson distributed with a rate of 100 cars per hour. Find the time required for more than 200 cars to have entered the parking lot with probability 0.90 (*a*) by using formula (*4.140*), and (*b*) by formula (*4.143*).

*Ans.* (*a*) 2.189 h (*b*) 2.1946 h

# Chapter 5

# Random Processes

## 5.1 INTRODUCTION

In this chapter, we introduce the concept of a random (or stochastic) process. The theory of random processes was first developed in connection with the study of fluctuations and noise in physical systems. A random process is the mathematical model of an empirical process whose development is governed by probability laws. Random processes provides useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth, and management sciences.

## 5.2 RANDOM PROCESSES

### 1. Defintion:

A *random process* is a family of r.v.'s $\{X(t), t \in T\}$ defined on a given probability space, indexed by the parameter $t$, where $t$ varies over an index set $T$.

Recall that a random variable is a function defined on the sample space $S$ (Sec. 2.2). Thus, a random process $\{X(t), t \in T\}$ is really a function of two arguments $\{X(t, \zeta), t \in T, \zeta \in S\}$. For a fixed $t(=t_k)$, $X(t_k, \zeta) = X_k(\zeta)$ is a r.v. denoted by $X(t_k)$, as $\zeta$ varies over the sample space $S$. On the other hand, for a fixed sample point $\zeta_i \in S$, $X(t, \zeta_i) = X_i(t)$ is a single function of time $t$, called a *sample function* or a *realization* of the process. The totality of all sample functions is called an *ensemble*.

Of course if both $\zeta$ and $t$ are fixed, $X(t_k, \zeta_i)$ is simply a real number. In the following we use the notation $X(t)$ to represent $X(t, \zeta)$.

### B. Description of a Random Process:

In a random process $\{X(t), t \in T\}$, the index set $T$ is called the *parameter set* of the random process. The values assumed by $X(t)$ are called *states*, and the set of all possible values forms the *state space* $E$ of the random process. If the index set $T$ of a random process is discrete, then the process is called a *discrete-parameter* (or *discrete-time*) process. A discrete-parameter process is also called a *random sequence* and is denoted by $\{X_n, n = 1, 2, \ldots\}$. If $T$ is continuous, then we have a *continuous-parameter* (or *continuous-time*) process. If the state space $E$ of a random process is discrete, then the process is called a *discrete-state* process, often referred to as a *chain*. In this case, the state space $E$ is often assumed to be $\{0, 1, 2, \ldots\}$. If the state space $E$ is continuous, then we have a *continuous-state* process.

A complex random process $X(t)$ is defined by

$$X(t) = X_1(t) + jX_2(t)$$

where $X_1(t)$ and $X_2(t)$ are (real) random processes and $j = \sqrt{-1}$. Throughout this book, all random processes are real random processes unless specified otherwise.

## 5.3 CHARACTERIZATION OF RANDOM PROCESSES

### A. Probabilistic Descriptions:

Consider a random process $X(t)$. For a fixed time $t_1$, $X(t_1) = X_1$ is a r.v., and its cdf $F_X(x_1; t_1)$ is defined as

$$F_X(x_1; t_1) = P\{X(t_1) \leq x_1\} \tag{5.1}$$

$F_X(x_1; t_1)$ is known as the *first-order distribution* of $X(t)$. Similarly, given $t_1$ and $t_2$, $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two r.v.'s. Their joint distribution is known as the *second-order distribution* of $X(t)$ and is given by

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\} \tag{5.2}$$

In general, we define the *nth-order distribution* of $X(t)$ by

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) \le x_1, \ldots, X(t_n) \le x_n\} \tag{5.3}$$

If $X(t)$ is a discrete-time process, then $X(t)$ is specified by a collection of pmf's:

$$p_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) = x_1, \ldots, X(t_n) = x_n\} \tag{5.4}$$

If $X(t)$ is a continuous-time process, then $X(t)$ is specified by a collection of pdf's:

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{\partial x_1 \cdots \partial x_n} \tag{5.5}$$

The complete characterization of $X(t)$ requires knowledge of all the distributions as $n \to \infty$. Fortunately, often much less is sufficient.

**B. Mean, Correlation, and Covariance Functions:**

As in the case of r.v.'s, random processes are often described by using statistical averages. The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)] \tag{5.6}$$

where $X(t)$ is treated as a random variable for a fixed value of $t$. In general, $\mu_X(t)$ is a function of time, and it is often called the *ensemble average* of $X(t)$. A measure of dependence among the r.v.'s of $X(t)$ is provided by its *autocorrelation function*, defined by

$$R_X(t, s) = E[X(t)X(s)] \tag{5.7}$$

Note that

$$R_X(t, s) = R_X(s, t) \tag{5.8}$$

and

$$R_X(t, t) = E[X^2(t)] \tag{5.9}$$

The *autocovariance function* of $X(t)$ is defined by

$$\begin{aligned} K_X(t, s) = \mathrm{Cov}[X(t), X(s)] &= E\{[X(t) - \mu_X(t)][X(s) - \mu_X(s)]\} \\ &= R_X(t, s) - \mu_X(t)\mu_X(s) \end{aligned} \tag{5.10}$$

It is clear that if the mean of $X(t)$ is zero, then $K_X(t, s) = R_X(t, s)$. Note that the *variance* of $X(t)$ is given by

$$\sigma_X^2(t) = \mathrm{Var}[X(t)] = E\{[X(t) - \mu_X(t)]^2\} = K_X(t, t) \tag{5.11}$$

If $X(t)$ is a complex random process, then its autocorrelation function $R_X(t, s)$ and autocovariance function $K_X(t, s)$ are defined, respectively, by

$$R_X(t, s) = E[X(t)X^*(s)] \tag{5.12}$$

and

$$K_X(t, s) = E\{[X(t) - \mu_X(t)][X(s) - \mu_X(s)]^*\} \tag{5.13}$$

where * denotes the complex conjugate.

## 5.4 CLASSIFICATION OF RANDOM PROCESSES

If a random process $X(t)$ possesses some special probabilistic structure, we can specify less to characterize $X(t)$ completely. Some simple random processes are characterized completely by only the first- and second-order distributions.

### A. Stationary Processes:

A random process $\{X(t), t \in T\}$ is said to be *stationary* or *strict-sense stationary* if, for all $n$ and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n)$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau) \tag{5.14}$$

for any $\tau$. Hence, the distribution of a stationary process will be unaffected by a shift in the time origin, and $X(t)$ and $X(t + \tau)$ will have the same distributions for any $\tau$. Thus, for the first-order distribution,

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x) \tag{5.15}$$

and

$$f_X(x; t) = f_X(x) \tag{5.16}$$

Then

$$\mu_X(t) = E[X(t)] = \mu \tag{5.17}$$

$$\text{Var}[X(t)] = \sigma^2 \tag{5.18}$$

where $\mu$ and $\sigma^2$ are contants. Similarly, for the second-order distribution,

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1) \tag{5.19}$$

and

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1) \tag{5.20}$$

Nonstationary processes are characterized by distributions depending on the points $t_1, t_2, \ldots, t_n$.

### B. Wide-Sense Stationary Processes:

If stationary condition (5.14) of a random process $X(t)$ does not hold for all $n$ but holds for $n \leq k$, then we say that the process $X(t)$ is *stationary to order k*. If $X(t)$ is stationary to order 2, then $X(t)$ is said to be *wide-sense stationary* (WSS) or *weak stationary*. If $X(t)$ is a WSS random process, then we have

1. $E[X(t)] = \mu$ (constant) (5.21)
2. $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$ (5.22)

Note that a strict-sense stationary process is also a WSS process, but, in general, the converse is not true.

### C. Independent Processes:

In a random process $X(t)$, if $X(t_i)$ for $i = 1, 2, \ldots, n$ are independent r.v.'s, so that for $n = 2, 3, \ldots,$

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \prod_{i=1}^{n} F_X(x_i; t_i) \tag{5.23}$$

then we call $X(t)$ an *independent random process*. Thus, a first-order distribution is sufficient to characterize an independent random process $X(t)$.

### D. Processes with Stationary Independent Increments:

A random process $\{X(t), t \geq 0\}$ is said to have *independent increments* if whenever $0 < t_1 < t_2 < \cdots < t_n$,

$$X(0), X(t_1) - X(0), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$$

are independent. If $\{X(t), t \geq 0\}$ has independent increments and $X(t) - X(s)$ has the same distribution as $X(t+h) - X(s+h)$ for all $s, t, h \geq 0$, $s < t$, then the process $X(t)$ is said to have *stationary independent increments*.

Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments and assume that $X(0) = 0$. Then (Probs. 5.21 and 5.22)

$$E[X(t)] = \mu_1 t \tag{5.24}$$

where $\mu_1 = E[X(1)]$ and

$$\text{Var}[X(t)] = \sigma_1^2 t \tag{5.25}$$

where $\sigma_1^2 = \text{Var}[X(1)]$.

From Eq. (5.24), we see that processes with stationary independent increments are nonstationary. Examples of processes with stationary independent increments are Poisson processes and Wiener processes, which are discussed in later sections.

## E. Markov Processes:

A random process $\{X(t), t \in T\}$ is said to be a *Markov process* if

$$P\{X(t_{n+1}) \leq x_{n+1} \mid X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_n) = x_n\} = P\{X(t_{n+1}) \leq x_{n+1} \mid X(t_n) = x_n\} \tag{5.26}$$

whenever $t_1 < t_2 < \cdots < t_n < t_{n+1}$.

A discrete-state Markov process is called a *Markov chain*. For a discrete-parameter Markov chain $\{X_n, n \geq 0\}$ (see Sec. 5.5), we have for every $n$

$$P(X_{n+1} = j \mid X_0 = i_0, X_i = i_1, \ldots, X_n = i) = P(X_{n+1} = j \mid X_n = i) \tag{5.27}$$

Equation (5.26) or Eq. (5.27) is referred to as the *Markov property* (which is also known as the *memoryless property*). This property of a Markov process states that the future state of the process depends only on the present state and not on the past history. Clearly, any process with independent increments is a Markov process.

Using the Markov property, the $n$th-order distribution of a Markov process $X(t)$ can be expressed as (Prob. 5.25)

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1; t_1) \prod_{k=2}^{n} P\{X(t_k) \leq x_k\} \mid X(t_{k-1}) = x_{k-1}\} \tag{5.28}$$

Thus, all finite-order distributions of a Markov process can be expressed in terms of the second-order distributions.

## F. Normal Processes:

A random process $\{X(t), t \in T\}$ is said to be a *normal* (or *gaussian*) process if for any integer $n$ and any subset $\{t_1, \ldots, t_n\}$ of $T$, the $n$ r.v.'s $X(t_1), \ldots, X(t_n)$ are jointly normally distributed in the sense that their joint characteristic function is given by

$$\begin{aligned} \Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) &= E\{\exp j[\omega_1 X(t_1) + \cdots + \omega_n X(t_n)]\} \\ &= \exp\left\{ j \sum_{i=1}^{n} \omega_i E[X(t_i)] - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_i \omega_k \text{Cov}[X(t_i), X(t_k)] \right\} \end{aligned} \tag{5.29}$$

where $\omega_1, \ldots, \omega_n$ are any real numbers (see Probs. 5.59 and 5.60). Equation (5.29) shows that a normal process is completely characterized by the second-order distributions. Thus, if a normal process is wide-sense stationary, then it is also strictly stationary.

### G. Ergodic Processes:

Consider a random process $\{X(t), -\infty < t < \infty\}$ with a typical sample function $x(t)$. The time average of $x(t)$ is defined as

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt \tag{5.30}$$

Similarly, the time autocorrelation function $\bar{R}_X(\tau)$ of $x(t)$ is defined as

$$\bar{R}_X(\tau) = \langle x(t)x(t+\tau) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt \tag{5.31}$$

A random process is said to be *ergodic* if it has the property that the time averages of sample functions of the process are equal to the corresponding statistical or ensemble averages. The subject of *ergodicity* is extremely complicated. However, in most physical applications, it is assumed that stationary processes are ergodic.

## 5.5 DISCRETE-PARAMETER MARKOV CHAINS

In this section we treat a discrete-parameter Markov chain $\{X_n, n \geq 0\}$ with a discrete state space $E = \{0, 1, 2, \ldots\}$, where this set may be finite or infinite. If $X_n = i$, then the Markov chain is said to be in state $i$ at time $n$ (or the $n$th step). A discrete-parameter Markov chain $\{X_n, n \geq 0\}$ is characterized by [Eq. (5.27)]

$$P(X_{n+1} = j \mid X_0 = i_0, X_i = i_1, \ldots, X_n = i) = P(X_{n+1} = j \mid X_n = i) \tag{5.32}$$

where $P\{x_{n+1} = j \mid X_n = i\}$ are known as one-step transition probabilities. If $P\{x_{n+1} = j \mid X_n = i\}$ is independent of $n$, then the Markov chain is said to possess *stationary transition probabilities* and the process is referred to as a *homogeneous* Markov chain. Otherwise the process is known as a *nonhomogeneous* Markov chain. Note that the concepts of a Markov chain's having stationary transition probabilities and being a stationary random process should not be confused. The Markov process, in general, is not stationary. We shall consider only homogeneous Markov chains in this section.

### A. Transition Probability Matrix:

Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain with a discrete infinite state space $E = \{0, 1, 2, \ldots\}$. Then

$$p_{ij} = P\{X_{n+1} = j \mid X_n = i\} \qquad i \geq 0, j \geq 0 \tag{5.33}$$

regardless of the value of $n$. A *transition probability matrix of* $\{X_n, n \geq 0\}$ is defined by

$$P = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

where the elements satisfy

$$p_{ij} \geq 0 \qquad \sum_{j=0}^{\infty} p_{ij} = 1 \qquad i = 0, 1, 2, \ldots \tag{5.34}$$

In the case where the state space $E$ is finite and equal to $\{1, 2, \ldots, m\}$, $P$ is $m \times m$ dimensional; that is,

$$P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{21} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & P_{mm} \end{bmatrix}$$

where $$p_{ij} \geq 0 \qquad \sum_{j=1}^{m} p_{ij} = 1 \qquad i = 1, 2, \ldots, m \tag{5.35}$$

A square matrix whose elements satisfy Eq. (5.34) or (5.35) is called a *Markov* matrix or *stochastic* matrix.

**B. Higher-Order Transition Probabilities—Chapman-Kolmogorov Equation:**

Tractability of Markov chain models is based on the fact that the probability distribution of $\{X_n, n \geq 0\}$ can be computed by matrix manipulations.

Let $P = [p_{ij}]$ be the transition probability matrix of a Markov chain $\{X_n, n \geq 0\}$. Matrix powers of $P$ are defined by

$$P^2 = PP$$

with the $(i, j)$th element given by

$$p_{ij}^{(2)} = \sum_k p_{ik} p_{kj}$$

Note that when the state space $E$ is infinite, the series above converges, since by Eq. (5.34),

$$\sum_k p_{ik} p_{kj} \leq \sum_k p_{ik} = 1$$

Similarly, $P^3 = PP^2$ has the $(i, j)$th element

$$p_{ij}^{(3)} = \sum_k p_{ik} p_{kj}^{(2)}$$

and in general, $P^{n+1} = PP^n$ has the $(i, j)$th element

$$p_{ij}^{(n+1)} = \sum_k p_{ik} p_{kj}^{(n)} \tag{5.36}$$

Finally, we define $P^0 = I$, where $I$ is the identity matrix.

The $n$-step transition probabilities for the homogeneous Markov chain $\{X_n, n \geq 0\}$ are defined by

$$P(X_n = j \mid X_0 = i)$$

Then we can show that (Prob. 5.70)

$$p_{ij}^{(n)} = P(X_n = j \mid X_0 = i) \tag{5.37}$$

We compute $p_{ij}^{(n)}$ by taking matrix powers.

The matrix identity

$$P^{n+m} = P^n P^m \qquad n, m \geq 0$$

when written in terms of elements

$$p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)} \tag{5.38}$$

is known as the *Chapman-Kolmogorov equation*. It expresses the fact that a transition from $i$ to $j$ in $n + m$ steps can be achieved by moving from $i$ to an intermediate $k$ in $n$ steps (with probability $p_{ik}^{(n)}$), and then proceeding to $j$ from $k$ in $m$ steps (with probability $p_{kj}^{(m)}$). Furthermore, the events "go from $i$ to $k$ in $n$ steps" and "go from $k$ to $j$ in $m$ steps" are independent. Hence the probability of the transition from $i$ to $j$ in $n + m$ steps via $i, k, j$ is $p_{ik}^{(n)} p_{kj}^{(m)}$. Finally, the probability of the transition from $i$ to $j$ is obtained by summing over the intermediate state $k$.

### C. The Probability Distribution of $\{X_n, n \geq 0\}$:

Let $p_i(n) = P(X_n = i)$ and

$$\mathbf{p}(n) = [p_0(n) \quad p_1(n) \quad p_2(n) \quad \cdots]$$

where

$$\sum_k p_k(n) = 1$$

Then $p_i(0) = P(X_0 = i)$ are the *initial-state* probabilities,

$$\mathbf{p}(0) = [p_0(0) \quad p_1(0) \quad p_2(0) \quad \cdots]$$

is called the *initial-state probability vector*, and $\mathbf{p}(n)$ is called the *state probability vector after n transitions* or the *probability distribution* of $X_n$. Now it can be shown that (Prob. 5.29)

$$\mathbf{p}(n) = \mathbf{p}(0)P^n \tag{5.39}$$

which indicates that the probability distribution of a homogeneous Markov chain is completely determined by the one-step transition probability matrix $P$ and the initial-state probability vector $\mathbf{p}(0)$.

### D. Classification of States:

#### 1. *Accessible States:*

State $j$ is said to be *accessible* from state $i$ if for some $n \geq 0$, $p_{ij}^{(n)} > 0$, and we write $i \to j$. Two states $i$ and $j$ accessible to each other are said to *communicate*, and we write $i \leftrightarrow j$. If all states communicate with each other, then we say that the Markov chain is *irreducible*.

#### 2. *Recurrent States:*

Let $T_j$ be the time (or the number of steps) of the first visit to state $j$ after time zero, unless state $j$ is never visited, in which case we set $T_j = \infty$. Then $T_j$ is a discrete r.v. taking values in $\{1, 2, \ldots, \infty\}$. Let

$$f_{ij}^{(m)} = P(T_j = m \mid X_0 = i) = P(X_m = j, X_k \neq j, k = 1, 2, \ldots, m - 1 \mid X_0 = i) \tag{5.40}$$

and $f_{ij}^{(0)} = 0$ since $T_j \geq 1$. Then

$$f_{ij}^{(1)} = P(T_j = 1 \mid X_0 = i) = P(X_1 = j \mid X_0 = i) = p_{ij} \tag{5.41}$$

and

$$f_{ij}^{(m)} = \sum_{k \neq j} p_{ik} f_{kj}^{(m-1)} \qquad m = 2, 3, \ldots \tag{5.42}$$

The probability of visiting $j$ in finite time, starting from $i$, is given by

$$f_{ij} \equiv \sum_{n=0}^{\infty} f_{ij}^{(n)} = P(T_j < \infty \mid X_0 = i) \tag{5.43}$$

Now state $j$ is said to be *recurrent* if

$$f_{jj} = P(T_j < \infty \mid X_0 = j) = 1 \tag{5.44}$$

That is, starting from $j$, the probability of eventual return to $j$ is one. A recurrent state $j$ is said to be *positive recurrent* if

$$E(T_j \mid X_0 = j) < \infty \tag{5.45}$$

and state $j$ is said to be *null recurrent* if

$$E(T_j \mid X_0 = j) = \infty \tag{5.46}$$

Note that

$$E(T_j \mid X_0 = j) = \sum_{n=0}^{\infty} n f_{jj}^{(n)} \tag{5.47}$$

### 3. *Transient States:*

State $j$ is said to be *transient* (or *nonrecurrent*) if

$$f_{jj} = P(T_j < \infty \mid X_0 = j) < 1 \tag{5.48}$$

In this case there is positive probability of never returning to state $j$.

### 4. *Periodic and Aperiodic States:*

We define the period of state $j$ to be

$$d(j) = \gcd\{n \geq 1 : p_{jj}^{(n)} > 0\}$$

where gcd stands for greatest common divisor.

If $d(j) > 1$, then state $j$ is called *periodic* with period $d(j)$. If $d(j) = 1$, then state $j$ is called *aperiodic*. Note that whenever $p_{jj} > 0$, $j$ is aperiodic.

### 5. *Absorbing States:*

State $j$ is said to be an *absorbing state* if $p_{jj} = 1$; that is, once state $j$ is reached, it is never left.

## E. Absorption Probabilities:

Consider a Markov chain $X(n) = \{X_n, n \geq 0\}$ with finite state space $E = \{1, 2, \ldots, N\}$ and transition probability matrix $P$. Let $A = \{1, \ldots, m\}$ be the set of absorbing states and $B = \{m + 1, \ldots, N\}$ be a set of nonabsorbing states. Then the transition probability matrix $P$ can be expressed as

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ 0 & \cdot & \cdots & 1 & 0 & \cdots & 0 \\ p_{m+1,1} & \cdot & \cdots & p_{m+1,m} & p_{m+1,m+1} & \cdots & p_{m+1,N} \\ \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots\vdots & \vdots \\ p_{N,1} & \cdot & \cdots & p_{N,m} & p_{N,m+1} & \cdots & p_{N,N} \end{bmatrix} = \begin{bmatrix} I & O \\ R & Q \end{bmatrix} \tag{5.49a}$$

where $I$ is an $m \times m$ identity matrix, $O$ is an $m \times (N - m)$ zero matrix, and

$$R = \begin{bmatrix} p_{m+1,1} & \cdots & p_{m+1,m} \\ \vdots & \vdots\vdots & \vdots \\ p_{N,1} & \cdots & p_{N,m} \end{bmatrix} \qquad Q = \begin{bmatrix} p_{m+1,m+1} & \cdots & p_{m+1,N} \\ \vdots & \vdots\vdots & \vdots \\ p_{N,m+1} & \cdots & p_{N,N} \end{bmatrix} \tag{5.49b}$$

Note that the elements of $R$ are the one-step transition probabilities from nonabsorbing to absorbing states, and the elements of $Q$ are the one-step transition probabilities among the nonabsorbing states.

Let $U = [u_{kj}]$, where

$$u_{kj} = P\{X_n = j(\in A) \mid X_0 = k(\in B)\}$$

It is seen that $U$ is an $(N - m) \times m$ matrix and its elements are the absorption probabilities for the various absorbing states. Then it can be shown that (Prob. 5.40)

$$U = (I - Q)^{-1}R = \Phi R \tag{5.50}$$

The matrix $\Phi = (I - Q)^{-1}$ is known as the *fundamental matrix* of the Markov chain $X(n)$. Let $T_k$ denote the total time units (or steps) to absorption from state $k$. Let

$$\mathbf{T} = [T_{m+1} \quad T_{m+2} \quad \cdots \quad T_N]$$

Then it can be shown that (Prob. 5.74)

$$E(T_k) = \sum_{i=m+1}^{N} \phi_{ki} \qquad k = m + 1, \ldots, N \tag{5.51}$$

where $\phi_{ki}$ is the $(k, i)$th element of the fundamental matrix $\Phi$.

### F. Stationary Distributions:

Let $P$ be the transition probability matrix of a homogeneous Markov chain $\{X_n, n \geq 0\}$. If there exists a probability vector $\hat{\mathbf{p}}$ such that

$$\hat{\mathbf{p}}P = \hat{\mathbf{p}} \tag{5.52}$$

then $\hat{\mathbf{p}}$ is called a *stationary distribution* for the Markov chain. Equation (5.52) indicates that a stationary distribution $\hat{\mathbf{p}}$ is a (left) *eigenvector* of $P$ with *eigenvalue* 1. Note that any nonzero multiple of $\hat{\mathbf{p}}$ is also an eigenvector of $P$. But the stationary distribution $\hat{\mathbf{p}}$ is fixed by being a probability vector; that is, its components sum to unity.

### G. Limiting Distributions:

A Markov chain is called *regular* if there is a finite positive integer $m$ such that after $m$ time-steps, every state has a nonzero chance of being occupied, no matter what the initial state. Let $A > O$ denote that every element $a_{ij}$ of $A$ satisfies the condition $a_{ij} > 0$. Then, for a regular Markov chain with transition probability matrix $P$, there exists an $m > 0$ such that $P^m > O$. For a regular homogeneous Markov chain we have the following theorem:

**THEOREM 5.5.1**

Let $\{X_n, n \geq 0\}$ be a regular homogeneous finite-state Markov chain with transition matrix $P$. Then

$$\lim_{n \to \infty} P^n = \hat{P} \tag{5.53}$$

where $\hat{P}$ is a matrix whose rows are identical and equal to the stationary distribution $\hat{\mathbf{p}}$ for the Markov chain defined by Eq. (5.52).

## 5.6 POISSON PROCESSES

### A. Definitions:

Let $t$ represent a time variable. Suppose an experiment begins at $t = 0$. Events of a particular kind occur randomly, the first at $T_1$, the second at $T_2$, and so on. The r.v. $T_i$ denotes the time at which the $i$th event occurs, and the values $t_i$ of $T_i$ $(i = 1, 2, \ldots)$ are called *points of occurrence* (Fig. 5-1).

Fig. 5-1

Let $$Z_n = T_n - T_{n-1} \tag{5.54}$$

and $T_0 = 0$. Then $Z_n$ denotes the time between the $(n-1)$st and the $n$th events (Fig. 5-1). The sequence of ordered r.v.'s $\{Z_n, n \geq 1\}$ is sometimes called an *interarrival process.* If all r.v.'s $Z_n$ are independent and identically distributed, then $\{Z_n, n \geq 1\}$ is called a *renewal process* or a *recurrent process.* From Eq. (*5.54*), we see that

$$T_n = Z_1 + Z_2 + \cdots + Z_n$$

where $T_n$ denotes the time from the beginning until the occurrence of the $n$th event. Thus, $\{T_n, n \geq 0\}$ is sometimes called an *arrival process.*

## B. Counting Processes:

A random process $\{X(t), t \geq 0\}$ is said to be a *counting process* if $X(t)$ represents the total number of "events" that have occurred in the interval $(0, t)$. From its definition, we see that for a counting process, $X(t)$ must satisfy the following conditions:

1. $X(t) \geq 0$ and $X(0) = 0$.
2. $X(t)$ is integer valued.
3. $X(s) \leq X(t)$ if $s < t$.
4. $X(t) - X(s)$ equals the number of events that have occurred on the interval $(s, t)$.

A typical sample function (or realization) of $X(t)$ is shown in Fig. 5-2.

A counting process $X(t)$ is said to possess independent increments if the numbers of events which occur in disjoint time intervals are independent. A counting process $X(t)$ is said to possess stationary increments if the number of events in the interval $(s + h, t + h)$—that is, $X(t + h) - X(s + h)$—has the same distribution as the number of events in the interval $(s, t)$—that is, $X(t) - X(s)$—for all $s < t$ and $h > 0$.

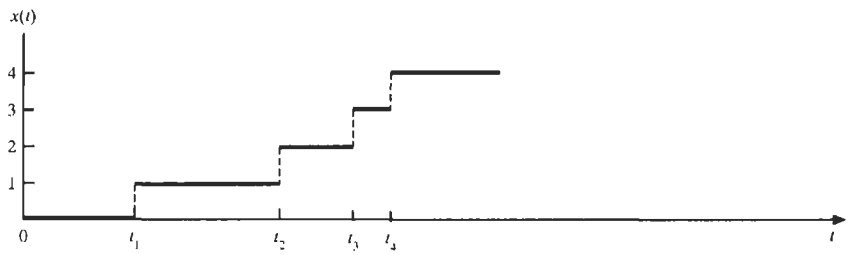



Fig. 5-2 A sample function of a counting process.

## C. Poisson Processes:

One of the most important types of counting processes is the *Poisson process* (or *Poisson counting process*), which is defined as follows:

**DEFINITION 5.6.1**

A counting process $X(t)$ is said to be a Poisson process with *rate* (or *intensity*) $\lambda(>0)$ if

1. $X(0) = 0$.
2. $X(t)$ has independent increments.
3. The number of events in any interval of length $t$ is Poisson distributed with mean $\lambda t$; that is, for all $s, t > 0$,

$$P[X(t+s) - X(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n = 0, 1, 2, \ldots \tag{5.55}$$

It follows from condition 3 of Def. 5.6.1 that a Poisson process has stationary increments and that

$$E[X(t)] = \lambda t \tag{5.56}$$

Then by Eq. (*2.43*) (Sec. 2.7C), we have

$$\mathrm{Var}[X(t)] = \lambda t \tag{5.57}$$

Thus, the expected number of events in the unit interval (0, 1), or any other interval of unit length, is just $\lambda$ (hence the name of the rate or intensity).

An alternative definition of a Poisson process is given as follows:

**DEFINITION 5.6.2**

A counting process $X(t)$ is said to be a Poisson process with rate (or intensity) $\lambda(>0)$ if

1. $X(0) = 0$.
2. $X(t)$ has independent and stationary increments.
3. $P[X(t+\Delta t) - X(t) = 1] = \lambda\,\Delta t + o(\Delta t)$
4. $P[X(t+\Delta t) - X(t) \geq 2] = o(\Delta t)$

where $o(\Delta t)$ is a function of $\Delta t$ which goes to zero faster than does $\Delta t$; that is,

$$\lim_{\Delta t \to 0}\frac{o(\Delta t)}{\Delta t} = 0 \tag{5.58}$$

***Note:*** Since addition or multiplication by a scalar does not change the property of approaching zero, even when divided by $\Delta t$, $o(\Delta t)$ satisfies useful identities such as $o(\Delta t) + o(\Delta t) = o(\Delta t)$ and $ao(\Delta t) = o(\Delta t)$ for all constant $a$.

It can be shown that Def. 5.6.1 and Def. 5.6.2 are equivalent (Prob. 5.49). Note that from conditions 3 and 4 of Def. 5.6.2, we have (Prob. 5.50)

$$P[X(t+\Delta t) - X(t) = 0] = 1 - \lambda\,\Delta t + o(\Delta t) \tag{5.59}$$

Equation (*5.59*) states that the probability that no event occurs in any short interval approaches unity as the duration of the interval approaches zero. It can be shown that in the Poisson process, the intervals between successive events are independent and identically distributed exponential r.v.'s (Prob. 5.53). Thus, we also identify the Poisson process as a renewal process with exponentially distributed intervals.

The autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Poisson process $X(t)$ with rate $\lambda$ are given by (Prob. 5.52)

$$R_X(t, s) = \lambda \min(t, s) + \lambda^2 ts \tag{5.60}$$

$$K_X(t, s) = \lambda \min(t, s) \tag{5.61}$$

## 5.7 WIENER PROCESSES

Another example of random processes with independent stationary increments is a *Wiener process.*

**DEFINITION 5.7.1**

A random process $\{X(t), t \geq 0\}$ is called a Wiener process if

1. $X(t)$ has stationary independent increments.
2. The increment $X(t) - X(s)$ $(t > s)$ is normally distributed.
3. $E[X(t)] = 0$.
4. $X(0) = 0$.

The Wiener process is also known as the *Brownian motion process,* since it originates as a model for Brownian motion, the motion of particles suspended in a fluid. From Def. 5.7.1, we can verify that a Wiener process is a normal process (Prob. 5.61) and

$$E[X(t)] = 0 \tag{5.62}$$

$$\text{Var}[X(t)] = \sigma^2 t \tag{5.63}$$

where $\sigma^2$ is a parameter of the Wiener process which must be determined from observations. When $\sigma^2 = 1$, $X(t)$ is called a *standard* Wiener (or standard Brownian motion) process.

The autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Wiener process $X(t)$ are given by (see Prob. 5.23)

$$R_X(t, s) = K_X(t, s) = \sigma^2 \min(t, s) \qquad s, t \geq 0 \tag{5.64}$$

**DEFINITION 5.7.2**

A random process $\{X(t), t \geq 0\}$ is called a *Wiener process with drift coefficient* $\mu$ if

1. $X(t)$ has stationary independent increments.
2. $X(t)$ is normally distributed with mean $\mu t$.
3. $X(0) = 0$.

From condition 2, the pdf of a standard Wiener process with drift coefficient $\mu$ is given by

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-\mu t)^2/(2t)} \tag{5.65}$$

# Solved Problems

## RANDOM PROCESSES

**5.1.** Let $X_1, X_2, \ldots$ be independent Bernoulli r.v.'s (Sec. 2.7A) with $P(X_n = 1) = p$ and $P(X_n = 0) = q = 1 - p$ for all $n$. The collection of r.v.'s $\{X_n, n \geq 1\}$ is a random process, and it is called a *Bernoulli process.*

(*a*) Describe the Bernoulli process.

(*b*) Construct a typical sample sequence of the Bernoulli process.

(*a*) The Bernoulli process $\{X_n, n \geq 1\}$ is a discrete-parameter, discrete-state process. The state space is $E = \{0, 1\}$, and the index set is $T = \{1, 2, \ldots\}$.

(*b*) A sample sequence of the Bernoulli process can be obtained by tossing a coin consecutively. If a head appears, we assign 1, and if a tail appears, we assign 0. Thus, for instance,

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ··· |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Coin tossing | H | T | T | H | H | H | T | H | H | T | ··· |
| $x_n$ | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | ··· |

The sample sequence $\{x_n\}$ obtained above is plotted in Fig. 5-3.

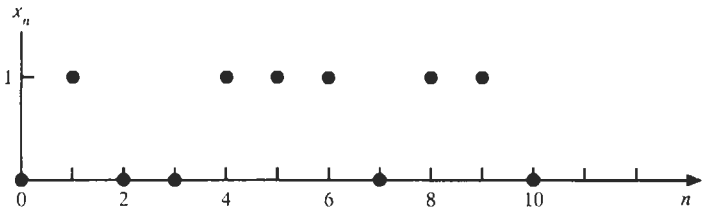



**Fig. 5-3** A sample function of a Bernoulli process.

**5.2.** Let $Z_1, Z_2, \ldots$ be independent identically distributed r.v.'s with $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all $n$. Let

$$X_n = \sum_{i=1}^{n} Z_i \qquad n = 1, 2, \ldots \tag{5.66}$$

and $X_0 = 0$. The collection of r.v.'s $\{X_n, n \geq 0\}$ is a random process, and it is called the *simple random walk* $X(n)$ in one dimension.

(*a*) Describe the simple random walk $X(n)$.

(*b*) Construct a typical sample sequence (or realization) of $X(n)$.

(*a*) The simple random walk $X(n)$ is a discrete-parameter (or time), discrete-state random process. The state space is $E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and the index parameter set is $T = \{0, 1, 2, \ldots\}$.

(*b*) A sample sequence $x(n)$ of a simple random walk $X(n)$ can be produced by tossing a coin every second and letting $x(n)$ increase by unity if a head appears and decrease by unity if a tail appears. Thus, for instance,

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | ··· |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Coin tossing | | H | T | T | H | H | H | T | H | H | T | ··· |
| $x(n)$ | 0 | 1 | 0 | −1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | ··· |

The sample sequence $x(n)$ obtained above is plotted in Fig. 5-4. The simple random walk $X(n)$ specified in this problem is said to be *unrestricted* because there are no bounds on the possible values of $X_n$.

The simple random walk process is often used in the following primitive gambling model: Toss a coin. If a head appears, you win one dollar; if a tail appears, you lose one dollar (see Prob. 5.38).

**5.3.** Let $\{X_n, n \geq 0\}$ be a simple random walk of Prob. 5.2. Now let the random process $X(t)$ be defined by

$$X(t) = X_n \qquad n \leq t < n + 1$$

(*a*) Describe $X(t)$.

(*b*) Construct a typical sample function of $X(t)$.

(*a*) The random process $X(t)$ is a continuous-parameter (or time), discrete-state random process. The state space is $E = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, and the index parameter set is $T = \{t, t \geq 0\}$.

(*b*) A sample function $x(t)$ of $X(t)$ corresponding to Fig. 5-4 is shown in Fig. 5-5.

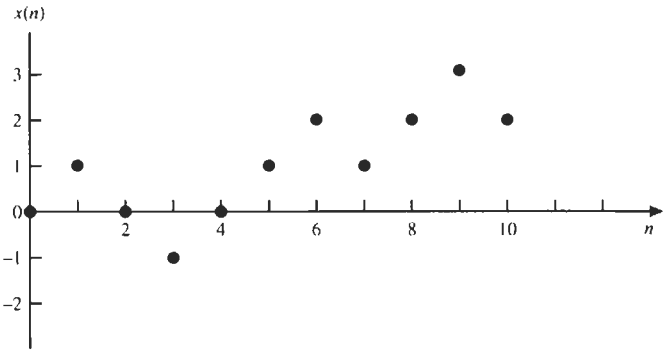



Fig. 5-4 A sample function of a random walk.

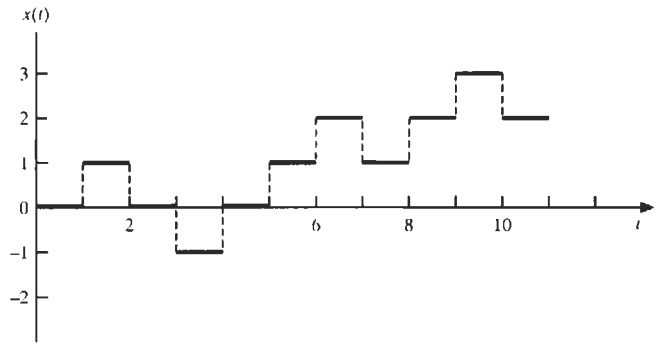



Fig. 5-5

**5.4.** Consider a random process $X(t)$ defined by

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

where $\omega$ is a constant and $Y$ is a uniform r.v. over (0, 1).

(*a*) Describe $X(t)$.

(*b*) Sketch a few typical sample functions of $X(t)$.

(*a*) The random process $X(t)$ is a continuous-parameter (or time), continuous-state random process. The state space is $E = \{x: -1 < x < 1\}$ and the index parameter set is $T = \{t: t \geq 0\}$.

(*b*) Three sample functions of $X(t)$ are sketched in Fig. 5-6.

**5.5.** Consider patients coming to a doctor's office at random points in time. Let $X_n$ denote the time (in hours) that the $n$th patient has to wait in the office before being admitted to see the doctor.

(*a*) Describe the random process $X(n) = \{X_n, n \geq 1\}$.

(*b*) Construct a typical sample function of $X(n)$.

(*a*) The random process $X(n)$ is a discrete-parameter, continuous-state random process. The state space is $E = \{x: x \geq 0\}$, and the index parameter set is $T = \{1, 2, \ldots\}$.

(*b*) A sample function $x(n)$ of $X(n)$ is shown in Fig. 5-7.

## CHARACTERIZATION OF RANDOM PROCESSES

**5.6.** Consider the Bernoulli process of Prob. 5.1. Determine the probability of occurrence of the sample sequence obtained in part (*b*) of Prob. 5.1.

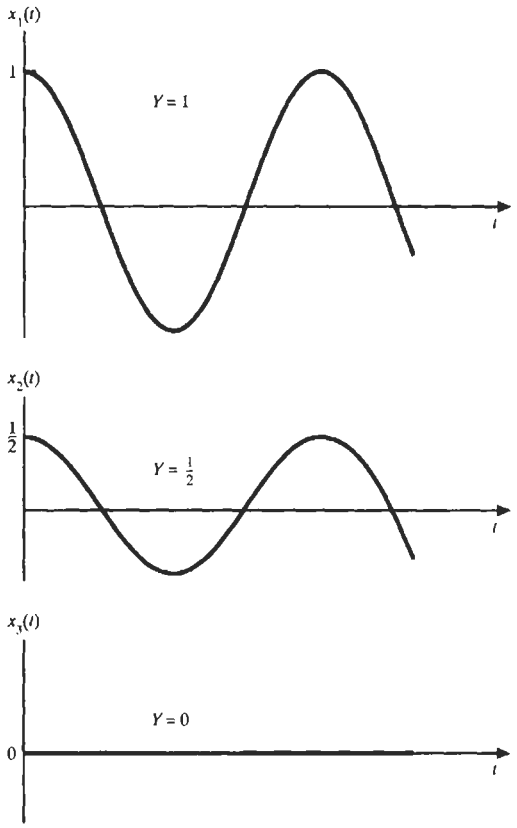



**Fig. 5-6**

Since $X_n$'s are independent, we have

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = P(X_1 = x_1)P(X_2 = x_2) \cdots P(X_n = x_n) \tag{5.67}$$

Thus, for the sample sequence of Fig. 5-3,

$$P(X_1 = 1, X_2 = 0, X_3 = 0, X_4 = 1, X_5 = 1, X_6 = 1, X_7 = 0, X_8 = 1, X_9 = 1, X_{10} = 0) = p^6q^4$$

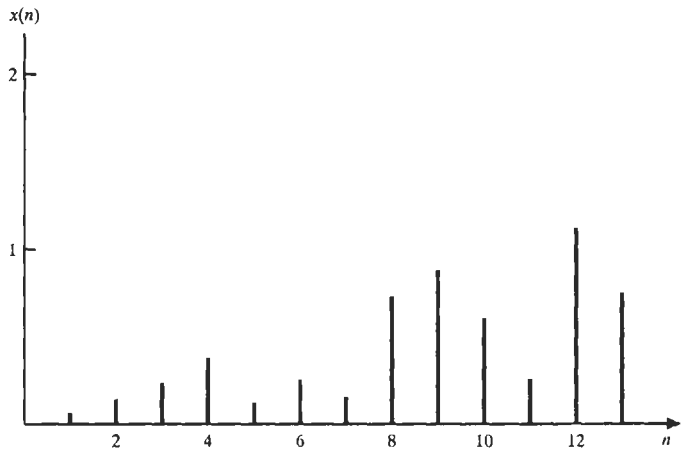



**Fig. 5-7**

**5.7.** Consider the random process $X(t)$ of Prob. 5.4. Determine the pdf's of $X(t)$ at $t = 0, \pi/4\omega, \pi/2\omega, \pi/\omega$.

For $t = 0$, $X(0) = Y \cos 0 = Y$. Thus,

$$f_{X(0)}(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $t = \pi/4\omega$, $X(\pi/4\omega) = Y \cos \pi/4 = 1/\sqrt{2}\, Y$. Thus,

$$f_{X(\pi/4\omega)}(x) = \begin{cases} \sqrt{2} & 0 < x < 1/\sqrt{2} \\ 0 & \text{otherwise} \end{cases}$$

For $t = \pi/2\omega$, $X(\pi/2\omega) = Y \cos \pi/2 = 0$; that is, $X(\pi/2\omega) = 0$ irrespective of the value of $Y$. Thus, the pmf of $X(\omega/2\omega)$ is

$$p_{X(\pi/2\omega)}(x) = P(X = 0) = 1$$

For $t = \pi/\omega$, $X(\pi/\omega) = Y \cos \pi = -Y$. Thus,

$$f_{X(\pi/\omega)}(x) = \begin{cases} 1 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases}$$

**5.8.** Derive the first-order probability distribution of the simple random walk $X(n)$ of Prob. 5.2.

The first-order probability distribution of the simple random walk $X(n)$ is given by

$$p_n(k) = P(X_n = k)$$

where $k$ is an integer. Note that $P(X_0 = 0) = 1$. We note that $p_n(k) = 0$ if $n < |k|$ because the simple random walk cannot get to level $k$ in less than $|k|$ steps. Thus, $n \geq |k|$.

Let $N_n^+$ and $N_n^-$ be the r.v.'s denoting the numbers of $+1$s and $-1$s, respectively, in the first $n$ steps. Then

$$n = N_n^+ + N_n^- \tag{5.68}$$

$$X_n = N_n^+ - N_n^- \tag{5.69}$$

Adding Eqs. (5.68) and (5.69), we get

$$N_n^+ = \tfrac{1}{2}(n + X_n) \tag{5.70}$$

Thus, $X_n = k$ if and only if $N_n^+ = \frac{1}{2}(n + k)$. From Eq. (5.70), we note that $2N_n^+ = n + X_n$ must be even. Thus, $X_n$ must be even if $n$ is even, and $X_n$ must be odd if $n$ is odd. We note that $N_n^+$ is a binomial r.v. with parameters $(n, p)$. Thus, by Eq. (2.36), we obtain

$$p_n(k) = \binom{n}{(n+k)/2} p^{(n+k)/2} q^{(n-k)/2} \qquad q = 1 - p \tag{5.71}$$

where $n \geq |k|$, and $n$ and $k$ are either both even or both odd.

**5.9.** Consider the simple random walk $X(n)$ of Prob. 5.2.

(*a*) Find the probability that $X(n) = -2$ after four steps.

(*b*) Verify the result of part (*a*) by enumerating all possible sample sequences that lead to the value $X(n) = -2$ after four steps.

(*a*) Setting $k = -2$ and $n = 4$ in Eq. (5.71), we obtain

$$P(X_4 = -2) = p_4(-2) = \binom{4}{1} pq^3 = 4pq^3 \qquad q = 1 - p$$

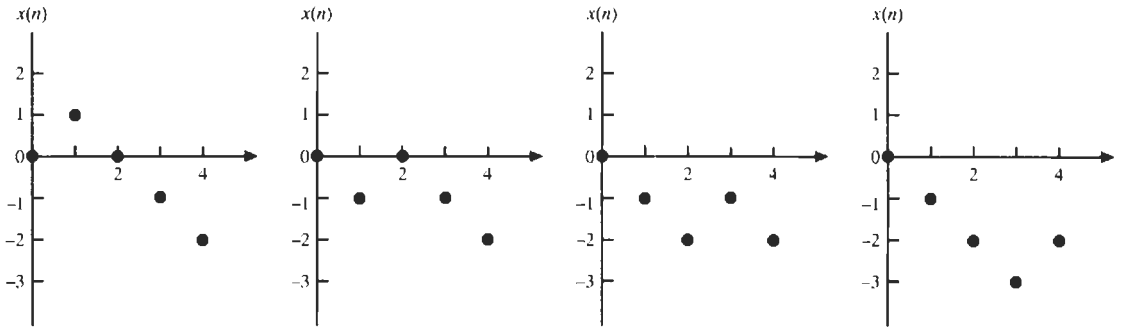



**Fig. 5-8**

(*b*) All possible sample functions that lead to the value $X_4 = -2$ after 4 steps are shown in Fig. 5-8. For each sample sequence, $P(X_4 = -2) = pq^3$. There are only four sample functions that lead to the value $X_4 = -2$ after four steps. Thus $P(X_4 = -2) = 4pq^3$.

**5.10** Find the mean and variance of the simple random walk $X(n)$ of Prob. 5.2.

From Eq. (*5.66*), we have

$$X_n = X_{n-1} + Z_n \qquad n = 1, 2, \ldots \tag{5.72}$$

and $X_0 = 0$ and $Z_n$ $(n = 1, 2, \ldots)$ are independent and identically distributed (iid) r.v.'s with

$$P(Z_n = +1) = p \qquad P(Z_n = -1) = q = 1 - p$$

From Eq. (*5.72*), we observe that

$$\begin{aligned} X_1 &= X_0 + Z_1 = Z_1 \\ X_2 &= X_1 + Z_2 = Z_1 + Z_2 \\ &\vdots \\ X_n &= Z_1 + Z_2 + \cdots + Z_n \end{aligned} \tag{5.73}$$

Then, because the $Z_n$ are iid r.v.'s and $X_0 = 0$, by Eqs. (*4.108*) and (*4.112*), we have

$$E(X_n) = E\left(\sum_{k=1}^{n} Z_k\right) = nE(Z_k)$$

$$\mathrm{Var}(X_n) = \mathrm{Var}\left(\sum_{k=1}^{n} Z_k\right) = n\,\mathrm{Var}(Z_k)$$

Now
$$E(Z_k) = (1)p + (-1)q = p - q \tag{5.74}$$

$$E(Z_k^2) = (1)^2 p + (-1)^2 q = p + q = 1 \tag{5.75}$$

Thus
$$\mathrm{Var}(Z_k) = E(Z_k^2) - [E(Z_k)]^2 = 1 - (p - q)^2 = 4pq \tag{5.76}$$

Hence,
$$E(X_n) = n(p - q) \qquad q = 1 - p \tag{5.77}$$

$$\mathrm{Var}(X_n) = 4npq \qquad q = 1 - p \tag{5.78}$$

Note that if $p = q = \frac{1}{2}$, then

$$E(X_n) = 0 \tag{5.79}$$

$$\mathrm{Var}(X_n) = n \tag{5.80}$$

**5.11.** Find the autocorrelation function $R_X(n, m)$ of the simple random walk $X(n)$ of Prob. 5.2.

From Eq. (5.73), we can express $X_n$ as

$$X_n = \sum_{i=0}^{n} Z_i \qquad n = 1, 2, \ldots \tag{5.81}$$

where $Z_0 = X_0 = 0$ and $Z_i$ $(i \geq 1)$ are iid r.v.'s with

$$P(Z_i = +1) = p \qquad P(Z_i = -1) = q = 1 - p$$

By Eq. (5.7),

$$R_X(n, m) = E[X(n)X(m)] = E(X_n X_m)$$

Then by Eq. (5.81),

$$R_X(n, m) = \sum_{i=0}^{n} \sum_{k=0}^{m} E(Z_i Z_k) = \sum_{i=0}^{\min(n, m)} E(Z_i^2) + \sum_{\substack{i=0 \\ i \neq k}}^{n} \sum_{k=0}^{m} E(Z_i)E(Z_k) \tag{5.82}$$

Using Eqs. (5.74) and (5.75), we obtain

$$R_X(n, m) = \min(n, m) + [nm - \min(n, m)](p - q)^2 \tag{5.83}$$

or

$$R_X(n, m) = \begin{cases} m + (nm - m)(p - q)^2 & m < n \\ n + (nm - n)(p - q)^2 & n < m \end{cases} \tag{5.84}$$

Note that if $p = q = \frac{1}{2}$, then

$$R_X(n, m) = \min(n, m) \qquad n, m > 0 \tag{5.85}$$

**5.12.** Consider the random process $X(t)$ of Prob. 5.4; that is,

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

where $\omega$ is a constant and $Y$ is a uniform r.v. over (0, 1).

(*a*) Find $E[X(t)]$.

(*b*) Find the autocorrelation function $R_X(t, s)$ of $X(t)$.

(*c*) Find the autocovariance function $K_X(t, s)$ of $X(t)$.

(*a*) From Eqs. (2.46) and (2.91), we have $E(Y) = \frac{1}{2}$ and $E(Y^2) = \frac{1}{3}$. Thus

$$E[X(t)] = E(Y \cos \omega t) = E(Y) \cos \omega t = \tfrac{1}{2} \cos \omega t \tag{5.86}$$

(*b*) By Eq. (5.7), we have

$$\begin{aligned} R_X(t, s) &= E[X(t)X(s)] = E(Y^2 \cos \omega t \cos \omega s) \\ &= E(Y^2) \cos \omega t \cos \omega s = \tfrac{1}{3} \cos \omega t \cos \omega s \end{aligned} \tag{5.87}$$

(*c*) By Eq. (5.10), we have

$$\begin{aligned} K_X(t, s) &= R_X(t, s) - E[X(t)]E[X(s)] \\ &= \tfrac{1}{3} \cos \omega t \cos \omega s - \tfrac{1}{4} \cos \omega t \cos \omega s \\ &= \tfrac{1}{12} \cos \omega t \cos \omega s \end{aligned} \tag{5.88}$$

**5.13.** Consider a discrete-parameter random process $X(n) = \{X_n, n \geq 1\}$ where the $X_n$'s are iid r.v.'s with common cdf $F_X(x)$, mean $\mu$, and variance $\sigma^2$.

(*a*) Find the joint cdf of $X(n)$.

(*b*) Find the mean of $X(n)$.

(*c*) Find the autocorrelation function $R_X(n, m)$ of $X(n)$.

(*d*) Find the autocovariance function $K_X(n, m)$ of $X(n)$.

(*a*) Since the $X_n$'s are iid r.v.'s with common cdf $F_X(x)$, the joint cdf of $X(n)$ is given by

$$F_X(x_1, \ldots, x_n) = \prod_{i=1}^{n} F_X(x_i) = [F_X(x)]^n \tag{5.89}$$

(*b*) The mean of $X(n)$ is

$$\mu_X(n) = E(X_n) = \mu \qquad \text{for all } n \tag{5.90}$$

(*c*) If $n \neq m$, by Eqs. (*5.7*) and (*5.90*),

$$R_X(n, m) = E(X_n X_m) = E(X_n)E(X_m) = \mu^2$$

If $n = m$, then by Eq. (*2.31*),

$$E(X_n^2) = \text{Var}(X_n) + [E(X_n)]^2 = \sigma^2 + \mu^2$$

Hence,

$$R_X(n, m) = \begin{cases} \mu^2 & n \neq m \\ \sigma^2 + \mu^2 & n = m \end{cases} \tag{5.91}$$

(*d*) By Eq. (*5.10*),

$$K_X(n, m) = R_X(n, m) - \mu_X(n)\mu_X(m) = \begin{cases} 0 & n \neq m \\ \sigma^2 & n = m \end{cases} \tag{5.92}$$

## CLASSIFICATION OF RANDOM PROCESSES

**5.14.** Show that a random process which is stationary to order $n$ is also stationary to all orders lower than $n$.

Assume that Eq. (*5.14*) holds for some particular $n$; that is,

$$P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\} = P\{X(t_1 + \tau) \leq x_1, \ldots, X(t_n + \tau) \leq x_n\}$$

for any $\tau$. Letting $x_n \to \infty$, we have [see Eq. (*3.63*)]

$$P\{X(t_1) \leq x_1, \ldots, X(t_{n-1}) \leq x_{n-1}\} = P\{X(t_1 + \tau) \leq x_1, \ldots, X(t_{n-1} + \tau) \leq x_{n-1}\}$$

and the process is stationary to order $n - 1$. Continuing the same procedure, we see that the process is stationary to all orders lower than $n$.

**5.15.** Show that if $\{X(t), t \in T\}$ is a strict-sense stationary random process, then it is also WSS.

Since $X(t)$ is strict-sense stationary, the first- and second-order distributions are invariant through time translation for all $\tau \in T$. Then we have

$$\mu_X(t) = E[X(t)] = E[X(t + \tau)] = \mu_X(t + \tau)$$

and hence the mean function $\mu_X(t)$ must be constant; that is,

$$E[X(t)] = \mu \text{ (constant)}$$

Similarly, we have

$$E[X(s)X(t)] = E[X(s + \tau)X(t + \tau)]$$

so that the autocorrelation function would depend on the time points $s$ and $t$ only through the difference $|t - s|$. Thus, $X(t)$ is WSS.

**5.16.** Let $\{X_n, n \geq 0\}$ be a sequence of iid r.v.'s with mean 0 and variance 1. Show that $\{X_n, n \geq 0\}$ is a WSS process.

By Eq. (*5.90*),

$$E(X_n) = 0 \text{ (constant)} \qquad \text{for all } n$$

and by Eq. (5.91),

$$R_X(n, n+k) = E(X_n X_{n+k}) = \begin{cases} E(X_n)E(X_{n+k}) = 0 & k \neq 0 \\ E(X_n^2) = \text{Var}(X_n) = 1 & k = 0 \end{cases}$$

which depends only on $k$. Thus, $\{X_n\}$ is a WSS process.

**5.17.** Show that if a random process $X(t)$ is WSS, then it must also be covariance stationary.

If $X(t)$ is WSS, then

$$E[X(t)] = \mu \text{ (constant)} \qquad \text{for all } t$$
$$R_X(t, t+\tau)] = R_X(\tau) \qquad \text{for all } t$$

Now
$$K_X(t, t+\tau) = \text{Cov}[X(t)X(t+\tau)] = R_X(t, t+\tau) - E[X(t)]E[X(t+\tau)]$$
$$= R_X(\tau) - \mu^2$$

which indicates that $K_X(t, t+\tau)$ depends only on $\tau$; thus, $X(t)$ is covariance stationary.

**5.18.** Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t \qquad -\infty < t < \infty \tag{5.93}$$

where $\omega$ is constant and $U$ and $V$ are r.v.'s.

(*a*) Show that the condition

$$E(U) = E(V) = 0 \tag{5.94}$$

is necessary for $X(t)$ to be stationary.

(*b*) Show that $X(t)$ is WSS if and only if $U$ and $V$ are uncorrelated with equal variance; that is,

$$E(UV) = 0 \qquad E(U^2) = E(V^2) = \sigma^2 \tag{5.95}$$

(*a*) Now

$$\mu_X(t) = E[X(t)] = E(U) \cos \omega t + E(V) \sin \omega t$$

must be independent of $t$ for $X(t)$ to be stationary. This is possible only if $\mu_X(t) = 0$, that is, $E(U) = E(V) = 0$.

(*b*) If $X(t)$ is WSS, then

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X^2$$

But $X(0) = U$ and $X(\pi/2\omega) = V$; thus

$$E(U^2) = E(V^2) = \sigma_X^2 = \sigma^2$$

Using the above result, we obtain

$$\begin{aligned} R_X(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\{(U \cos \omega t + V \sin \omega t)[U \cos \omega(t+\tau) + V \sin \omega(t+\tau)]\} \\ &= \sigma^2 \cos \omega\tau + E(UV) \sin(2\omega t + \omega\tau) \end{aligned} \tag{5.96}$$

which will be a function of $\tau$ only if $E(UV) = 0$. Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part (*a*) and Eq. (5.96), we have

$$\mu_X(t) = 0$$
$$R_X(t, t+\tau) = \sigma^2 \cos \omega\tau = R_X(\tau)$$

Hence, $X(t)$ is WSS.

**5.19.** Consider a random process $X(t)$ defined by

$$X(t) = U \cos t + V \sin t \qquad -\infty < t < \infty$$

where $U$ and $V$ are independent r.v.'s, each of which assumes the values $-2$ and 1 with the probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Show that $X(t)$ is WSS but not strict-sense stationary.

We have

$$E(U) = E(V) = \tfrac{1}{3}(-2) + \tfrac{2}{3}(1) = 0$$

$$E(U^2) = E(V^2) = \tfrac{1}{3}(-2)^2 + \tfrac{2}{3}(1)^2 = 2$$

Since $U$ and $V$ are independent,

$$E(UV) = E(U)E(V) = 0$$

Thus, by the results of Prob. 5.18, $X(t)$ is WSS. To see if $X(t)$ is strict-sense stationary, we consider $E[X^3(t)]$.

$$\begin{aligned} E[X^3(t)] &= E[(U \cos t + V \sin t)^3] \\ &= E(U^3)\cos^3 t + 3E(U^2V)\cos^2 t \sin t + 3E(UV^2)\cos t \sin^2 t + E(V^3)\sin^3 t \end{aligned}$$

Now

$$E(U^3) = E(V^3) = \tfrac{1}{3}(-2)^3 + \tfrac{2}{3}(1)^3 = -2$$

$$E(U^2V) = E(U^2)E(V) = 0 \qquad E(UV^2) = E(U)E(V^2) = 0$$

Thus

$$E[X^3(t)] = -2(\cos^3 t + \sin^3 t)$$

which is a function of $t$. From Eq. (*5.16*), we see that all the moments of a strict-sense stationary process must be independent of time. Thus $X(t)$ is not strict-sense stationary.

**5.20.** Consider a random process $X(t)$ defined by

$$X(t) = A \cos(\omega t + \Theta) \qquad -\infty < t < \infty$$

where $A$ and $\omega$ are constants and $\Theta$ is a uniform r.v. over $(-\pi, \pi)$. Show that $X(t)$ is WSS.

From Eq. (*2.44*), we have

$$f_\Theta(\theta) = \begin{cases} \dfrac{1}{2\pi} & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mu_X(t) = \frac{A}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0 \tag{5.97}$$

Setting $s = t + \tau$ in Eq. (5.7), we have

$$\begin{aligned} R_{XX}(t, t+\tau) &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\cos[\omega(t+\tau) + \theta)]\, d\theta \\ &= \frac{A^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}[\cos \omega\tau + \cos(2\omega t + 2\theta + \omega\tau)]\, d\theta \\ &= \frac{A^2}{2}\cos \omega\tau \end{aligned} \tag{5.98}$$

Since the mean of $X(t)$ is a constant and the autocorrelation of $X(t)$ is a function of time difference only, we conclude that $X(t)$ is WSS.

**5.21.** Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

$$E[X(t)] = \mu_1 t \tag{5.99}$$

where $\mu_1 = E[X(1)]$.

Let $$f(t) = E[X(t)] = E[X(t) - X(0)]$$

Then, for any $t$ and $s$ and using Eq. (*4.108*) and the property of the stationary independent increments, we have

$$\begin{aligned} f(t+s) &= E[X(t+s) - X(0)] \\ &= E[X(t+s) - X(s) + X(s) - X(0)] \\ &= E[X(t+s) - X(s)] + E[X(s) - X(0)] \\ &= E[X(t) - X(0)] + E[X(s) - X(0)] \\ &= f(t) + f(s) \end{aligned} \tag{5.100}$$

The only solution to the above functional equation is $f(t) = ct$, where $c$ is a constant. Since $c = f(1) = E[X(1)]$, we obtain

$$E[X(t)] = \mu_1 t \qquad \mu_1 = E[X(1)]$$

**5.22.** Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

(*a*) $$\mathrm{Var}[X(t)] = \sigma_1^2 t \tag{5.101}$$

(*b*) $$\mathrm{Var}[X(t) - X(s)] = \sigma_1^2 (t - s) \qquad t > s \tag{5.102}$$

where $\sigma_1^2 = \mathrm{Var}[X(1)]$.

(*a*) Let $$g(t) = \mathrm{Var}[X(t)] = \mathrm{Var}[X(t) - X(0)]$$

Then, for any $t$ and $s$ and using Eq. (*4.112*) and the property of the stationary independent increments, we get

$$\begin{aligned} g(t+s) &= \mathrm{Var}[X(t+s) - X(0)] \\ &= \mathrm{Var}[X(t+s) - X(s) + X(s) - X(0)] \\ &= \mathrm{Var}[X(t+s) - X(s)] + \mathrm{Var}[X(s) - X(0)] \\ &= \mathrm{Var}[X(t) - X(0)] + \mathrm{Var}[X(s) - X(0)] \\ &= g(t) + g(s) \end{aligned}$$

which is the same functional equation as Eq. (*5.100*). Thus, $g(t) = kt$, where $k$ is a constant. Since $k = g(1) = \mathrm{Var}[X(1)]$, we obtain

$$\mathrm{Var}[X(t)] = \sigma_1^2 t \qquad \sigma_1^2 = \mathrm{Var}[X(1)]$$

(*b*) Let $t > s$. Then

$$\begin{aligned} \mathrm{Var}[X(t)] &= \mathrm{Var}[X(t) - X(s) + X(s) - X(0)] \\ &= \mathrm{Var}[X(t) - X(s)] + \mathrm{Var}[X(s) - X(0)] \\ &= \mathrm{Var}[X(t) - X(s)] + \mathrm{Var}[X(s)] \end{aligned}$$

Thus, using Eq. (*5.101*), we obtain

$$\mathrm{Var}[X(t) - X(s)] = \mathrm{Var}[X(t)] - \mathrm{Var}[X(s)] = \sigma_1^2 (t - s)$$

**5.23.** Let $\{X(t), t \geq 0\}$ be a random process with stationary independent increments, and assume that $X(0) = 0$. Show that

$$\mathrm{Cov}[X(t), X(s)] = K_X(t, s) = \sigma_1^2 \min(t, s) \tag{5.103}$$

where $\sigma_1^2 = \mathrm{Var}[X(1)]$.

By definition (*2.28*),

$$\begin{aligned}\mathrm{Var}[X(t) - X(s)] &= E(\{X(t) - X(s) - E[X(t) - X(s)]\}^2)\\ &= E[(\{X(t) - E[X(t)]\} - \{X(s) - E[X(s)]\})^2]\\ &= E(\{X(t) - E[X(t)]\}^2 - 2\{X(t) - E[X(t)]\}\{X(s) - E[X(s)]\} + \{X(s) - E[X(s)]\}^2)\\ &= \mathrm{Var}[X(t)] - 2\,\mathrm{Cov}[X(t), X(s)] + \mathrm{Var}[X(s)]\end{aligned}$$

Thus, $$\mathrm{Cov}[X(t), X(s)] = \tfrac{1}{2}\{\mathrm{Var}[X(t)] + \mathrm{Var}[X(s)] - \mathrm{Var}[X(t) - X(s)]\}$$

Using Eqs. (*5.101*) and (*5.102*), we obtain

$$K_X(t, s) = \begin{cases}\frac{1}{2}\sigma_1^2[t + s - (t - s)] = \sigma_1^2 s & t > s\\ \frac{1}{2}\sigma_1^2[t + s - (s - t)] = \sigma_1^2 t & s > t\end{cases}$$

or $$K_X(t, s) = \sigma_1^2 \min(t, s)$$

where $\sigma_1^2 = \mathrm{Var}[X(1)]$.

**5.24.** (*a*) Show that a simple random walk $X(n)$ of Prob. 5.2 is a Markov chain.

(*b*) Find its one-step transition probabilities.

(*a*) From Eq. (*5.73*) (Prob. 5.10), $X(n) = \{X_n, n \geq 0\}$ can be expressed as

$$X_0 = 0 \qquad X_n = \sum_{i=1}^{n} Z_i \qquad n \geq 1$$

where $Z_n$ ($n = 1, 2, \ldots$) are iid r.v.'s with

$$P(Z_n = k) = a_k \qquad (k = 1, -1) \qquad \text{and} \qquad a_1 = p \qquad a_{-1} = q = 1 - p$$

Then $X(n) = \{X_n, n \geq 0\}$ is a Markov chain, since

$$\begin{aligned}&P(X_{n+1} = i_{n+1} \mid X_0 = 0, X_1 = i_1, \ldots, X_n = i_n)\\ &\quad = P(Z_{n+1} + i_n = i_{n+1} \mid X_0 = 0, X_1 = i_1, \ldots, X_n = i_n)\\ &\quad = P(Z_{n+1} = i_{n+1} - i_n) = a_{i_{n+1} - i_n} = P(X_{n+1} = i_{n+1} \mid X_n = i_n)\end{aligned}$$

since $Z_{n+1}$ is independent of $X_0, X_1, \ldots, X_n$.

(*b*) The one-step transition probabilities are given by

$$p_{jk} = P(X_n = k \mid X_{n-1} = j) = \begin{cases}p & k = j + 1\\ q = 1 - p & k = j - 1\\ 0 & \text{otherwise}\end{cases}$$

which do not depend on $n$. Thus, a simple random walk $X(n)$ is a homogeneous Markov chain.

**5.25.** Show that for a Markov process $X(t)$, the second-order distribution is sufficient to characterize $X(t)$.

Let $X(t)$ be a Markov process with the $n$th-order distribution

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\}$$

Then, using the Markov property (*5.26*), we have

$$\begin{aligned}F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) &= P\{X(t_n) \leq x_n \mid X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_{n-1}) \leq x_{n-1}\}\\ &\quad \times P\{X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_{n-1}) \leq x_{n-1}\}\\ &= P\{X(t_n) \leq x_n \mid X(t_{n-1}) \leq x_{n-1}\} F_X(x_1, \ldots, x_{n-1}; t_1, \ldots, t_{n-1})\end{aligned}$$

Applying the above relation repeatedly for lower-order distribution, we can write

$$F_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = F_X(x_1, t_1) \prod_{k=2}^{n} P\{X(t_k) \leq x_k \mid X(t_{k-1}) \leq x_{k-1}\} \tag{5.104}$$

Hence, all finite-order distributions of a Markov process can be completely determined by the second-order distribution.

**5.26.** Show that if a normal process is WSS, then it is also strict-sense stationary.

By Eq. (5.29), a normal random process $X(t)$ is completely characterized by the specification of the mean $E[X(t)]$ and the covariance function $K_X(t, s)$ of the process. Suppose that $X(t)$ is WSS. Then, by Eqs. (5.21) and (5.22), Eq. (5.29) becomes

$$\Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) = \exp\left\{ j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X(t_i - t_k)\omega_i \omega_k \right\} \tag{5.105}$$

Now we translate all of the time instants $t_1, t_2, \ldots, t_n$ by the same amount $\tau$. The joint characteristic function of the new r.v.'s $X(t_i + \tau)$, $i = 1, 2, \ldots, n$, is then

$$\begin{aligned} \Psi_{X(t_1+\tau)\cdots X(t_n+\tau)}(\omega_1, \ldots, \omega_n) &= \exp\left\{ j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X[t_i + \tau - (t_k + \tau)]\omega_i \omega_k \right\} \\ &= \exp\left\{ j \sum_{i=1}^{n} \mu\omega_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} K_X(t_i - t_k)\omega_i \omega_k \right\} \\ &= \Psi_{X(t_1)\cdots X(t_n)}(\omega_1, \ldots, \omega_n) \end{aligned} \tag{5.106}$$

which indicates that the joint characteristic function (and hence the corresponding joint pdf) is unaffected by a shift in the time origin. Since this result holds for any $n$ and any set of time instants ($t_i \in T$, $i = 1, 2, \ldots, n$), it follows that if a normal process is WSS, then it is also strict-sense stationary.

**5.27.** Let $\{X(t), -\infty < t < \infty\}$ be a zero-mean, stationary, normal process with the autocorrelation function

$$R_X(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & -T \le \tau \le T \\ 0 & \text{otherwise} \end{cases} \tag{5.107}$$

Let $\{X(t_i), i = 1, 2, \ldots, n\}$ be a sequence of $n$ samples of the process taken at the time instants

$$t_i = i\frac{T}{2} \qquad i = 1, 2, \ldots, n$$

Find the mean and the variance of the sample mean

$$\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} X(t_i) \tag{5.108}$$

Since $X(t)$ is zero-mean and stationary, we have

$$E[X(t_i)] = 0$$

and

$$R_X(t_i, t_k) = E[X(t_i)X(t_k)] = R_X(t_k - t_i) = R_X\left[(k - i)\frac{T}{2}\right]$$

Thus

$$E(\hat{\mu}_n) = E\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right] = \frac{1}{n} \sum_{i=1}^{n} E[X(t_i)] = 0 \tag{5.109}$$

and

$$\begin{aligned} \mathrm{Var}(\hat{\mu}_n) &= E\{[\hat{\mu}_n - E(\hat{\mu}_n)]^2\} = E(\hat{\mu}_n^2) \\ &= E\left\{\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right]\left[\frac{1}{n} \sum_{k=1}^{n} X(t_k)\right]\right\} \\ &= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} E[X(t_i)X(t_k)] = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} R_X\left[(k - i)\frac{T}{2}\right] \end{aligned}$$

By Eq. (*5.107*),

$$R_X[(k-i)T/2] = \begin{cases} 1 & k = i \\ \frac{1}{2} & |k-i| = 1 \\ 0 & |k-i| > 2 \end{cases}$$

Thus
$$\mathrm{Var}(\hat{\mu}_n) = \frac{1}{n^2}[n(1) + 2(n-1)(\tfrac{1}{2}) + 0] = \frac{1}{n^2}(2n-1) \tag{5.110}$$

## DISCRETE-PARAMETER MARKOV CHAINS

**5.28.** Show that if $P$ is a Markov matrix, then $P^n$ is also a Markov matrix for any positive integer $n$.

Let
$$P = [p_{ij}] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{12} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix}$$

Then by the property of a Markov matrix [Eq. (*5.35*)], we can write

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1m} \\ p_{12} & p_{22} & \cdots & p_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mm} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

or
$$P\mathbf{a} = \mathbf{a} \tag{5.111}$$

where
$$\mathbf{a}^T = [1 \quad 1 \quad \cdots \quad 1]$$

Premultiplying both sides of Eq. (*5.111*) by $P$, we obtain

$$P^2\mathbf{a} = P\mathbf{a} = \mathbf{a}$$

which indicates that $P^2$ is also a Markov matrix. Repeated premultiplication by $P$ yields

$$P^n\mathbf{a} = \mathbf{a}$$

which shows that $P^n$ is also a Markov matrix.

**5.29.** Verify Eq. (*5.39*); that is,

$$\mathbf{p}(n) = \mathbf{p}(0)P^n$$

We verify Eq. (*5.39*) by induction. If the state of $X_0$ is $i$, state $X_1$ will be $j$ only if a transition is made from $i$ to $j$. The events $\{X_0 = i, i = 1, 2, \ldots\}$ are mutually exclusive, and one of them must occur. Hence, by the law of total probability [Eq. (*1.44*)],

$$P(X_1 = j) = \sum_i P(X_0 = i)P(X_1 = j \mid X_0 = i)$$

or
$$p_j(1) = \sum_i p_i(0)p_{ij} \qquad j = 1, 2, \ldots \tag{5.112}$$

In terms of vectors and matrices, Eq. (*5.112*) can be expressed as

$$\mathbf{p}(1) = \mathbf{p}(0)P \tag{5.113}$$

Thus, Eq. (*5.39*) is true for $n = 1$. Assume now that Eq. (*5.39*) is true for $n = k$; that is,

$$\mathbf{p}(k) = \mathbf{p}(0)P^k$$

Again, by the law of total probability,

$$P(X_{k+1} = j) = \sum_i P(X_k = i)P(X_{k+1} = j \mid X_k = i)$$

or

$$p_j(k+1) = \sum_i p_i(k)p_{ij} \qquad j = 1, 2, \ldots \tag{5.114}$$

In terms of vectors and matrices, Eq. (5.114) can be expressed as

$$\mathbf{p}(k+1) = \mathbf{p}(k)P = \mathbf{p}(0)P^kP = \mathbf{p}(0)P^{k+1} \tag{5.115}$$

which indicates that Eq. (5.39) is true for $k+1$. Hence, we conclude that Eq. (5.39) is true for all $n \geq 1$.

**5.30.** Consider a two-state Markov chain with the transition probability matrix

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 < a < 1, 0 < b < 1 \tag{5.116}$$

(*a*) Show that the $n$-step transition probability matrix $P^n$ is given by

$$P^n = \frac{1}{a+b}\left\{\begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right\} \tag{5.117}$$

(*b*) Find $P^n$ when $n \to \infty$.

(*a*) From matrix analysis, the characteristic equation of $P$ is

$$c(\lambda) = |\lambda I - P| = \begin{vmatrix} \lambda - (1-a) & -a \\ -b & \lambda - (1-b) \end{vmatrix}$$
$$= (\lambda - 1)(\lambda - 1 + a + b) = 0$$

Thus, the eigenvalues of $P$ are $\lambda_1 = 1$ and $\lambda_2 = 1 - a - b$. Then, using the spectral decomposition method, $P^n$ can be expressed as

$$P^n = \lambda_1^n E_1 + \lambda_2^n E_2 \tag{5.118}$$

where $E_1$ and $E_2$ are constituent matrices of $P$, given by

$$E_1 = \frac{1}{\lambda_1 - \lambda_2}[P - \lambda_2 I] \qquad E_2 = \frac{1}{\lambda_2 - \lambda_1}[P - \lambda_1 I] \tag{5.119}$$

Substituting $\lambda_1 = 1$ and $\lambda_2 = 1 - a - b$ in the above expressions, we obtain

$$E_1 = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \qquad E_2 = \frac{1}{a+b}\begin{bmatrix} a & -a \\ -b & b \end{bmatrix}$$

Thus, by Eq. (5.118), we obtain

$$P^n = E_1 + (1-a-b)^n E_2$$
$$= \frac{1}{a+b}\left\{\begin{bmatrix} b & a \\ b & a \end{bmatrix} + (1-a-b)^n \begin{bmatrix} a & -a \\ -b & b \end{bmatrix}\right\} \tag{5.120}$$

(*b*) If $0 < a < 1, 0 < b < 1$, then $0 < 1 - a < 1$ and $|1 - a - b| < 1$. So $\lim_{n\to\infty}(1-a-b)^n = 0$ and

$$\lim_{n\to\infty} P^n = \frac{1}{a+b}\begin{bmatrix} b & a \\ b & a \end{bmatrix} \tag{5.121}$$

Note that a limiting matrix exists and has the same rows (see Prob. 5.47).

**5.31.** An example of a two-state Markov chain is provided by a communication network consisting of the sequence (or cascade) of stages of binary communication channels shown in Fig. 5-9. Here $X_n$

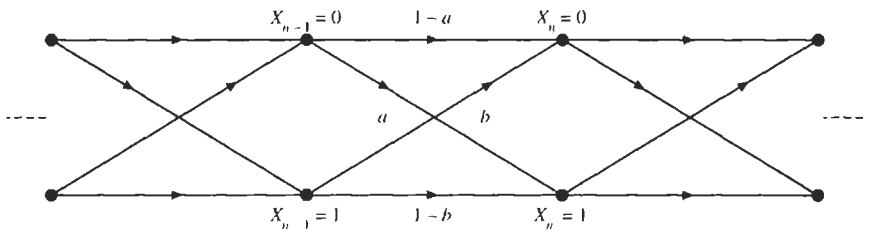



**Fig. 5-9** Binary communication network.

denotes the digit leaving the $n$th stage of the channel and $X_0$ denotes the digit entering the first stage. The transition probability matrix of this communication network is often called the *channel matrix* and is given by Eq. (*5.116*); that is,

$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \qquad 0 < a < 1,\ 0 < b < 1$$

Assume that $a = 0.1$ and $b = 0.2$, and the initial distribution is $P(X_0 = 0) = P(X_0 = 1) = 0.5$.

(*a*) Find the distribution of $X_n$.

(*b*) Find the distribution of $X_n$ when $n \to \infty$.

(*a*) The channel matrix of the communication network is

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

and the initial distribution is

$$\mathbf{p}(0) = [0.5 \quad 0.5]$$

By Eq. (*5.39*), the distribution of $X_n$ is given by

$$\mathbf{p}(n) = \mathbf{p}(0)P^n = [0.5 \quad 0.5]\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}^n$$

Letting $a = 0.1$ and $b = 0.2$ in Eq. (*5.117*), we get

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}^n = \frac{1}{0.3}\begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.1 \end{bmatrix} + \frac{(0.7)^n}{0.3}\begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2+(0.7)^n}{3} & \dfrac{1-(0.7)^n}{3} \\ \dfrac{2-2(0.7)^n}{3} & \dfrac{1+2(0.7)^n}{3} \end{bmatrix}$$

Thus, the distribution of $X_n$ is

$$\mathbf{p}(n) = [0.5 \quad 0.5]\begin{bmatrix} \dfrac{2+(0.7)^n}{3} & \dfrac{1-(0.7)^n}{3} \\ \dfrac{2-2(0.7)^n}{3} & \dfrac{1+2(0.7)^n}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2}{3} - \dfrac{(0.7)^n}{6} & \dfrac{1}{3} + \dfrac{(0.7)^n}{6} \end{bmatrix}$$

that is,

$$P(X_n = 0) = \frac{2}{3} - \frac{(0.7)^n}{6} \qquad \text{and} \qquad P(X_n = 1) = \frac{1}{3} + \frac{(0.7)^n}{6}$$

(*b*) Since $\lim_{n\to\infty}(0.7)^n = 0$, the distribution of $X_n$ when $n \to \infty$ is

$$P(X_\infty = 0) = \tfrac{2}{3} \qquad \text{and} \qquad P(X_\infty = 1) = \tfrac{1}{3}$$

**5.32.** Verify the transitivity property of the Markov chain; that is, if $i \to j$ and $j \to k$, then $i \to k$.

By definition, the relations $i \to j$ and $j \to k$ imply that there exist integers $n$ and $m$ such that $p_{ij}^{(n)} > 0$ and $p_{jk}^{(m)} > 0$. Then, by the Chapman-Kolmogorov equation (*5.38*), we have

$$p_{ik}^{(n+m)} = \sum_r p_{ir}^{(n)} p_{rk}^{(m)} \geq p_{ij}^{(n)} p_{jk}^{(m)} > 0 \tag{5.122}$$

Therefore $i \to k$.

**5.33.** Verify Eq. (*5.42*).

If the Markov chain $\{X_n\}$ goes from state $i$ to state $j$ in $m$ steps, the first step must take the chain from $i$ to some state $k$, where $k \neq j$. Now after that first step to $k$, we have $m - 1$ steps left, and the chain must get to state $j$, from state $k$, on the last of those steps. That is, the first visit to state $j$ must occur on the $(m - 1)$st step, starting now in state $k$. Thus we must have

$$f_{ij}^{(m)} = \sum_{k \neq j} p_{ik} f_{kj}^{(m-1)} \qquad m = 2, 3, \ldots$$

**5.34.** Show that in a finite-state Markov chain, not all states can be transient.

Suppose that the states are $0, 1, \ldots, m$, and suppose that they are all transient. Then by definition, after a finite amount of time (say $T_0$), state 0 will never be visited; after a finite amount of time (say $T_1$), state 1 will never be visited; and so on. Thus, after a finite time $T = \max\{T_0, T_1, \ldots, T_m\}$, no state will be visited. But as the process must be in some state after time $T$, we have a contradiction. Thus, we conclude that not all states can be transient and at least one of the states must be recurrent.

**5.35.** A state transition diagram of a finite-state Markov chain is a line diagram with a vertex corresponding to each state and a directed line between two vertices $i$ and $j$ if $p_{ij} > 0$. In such a diagram, if one can move from $i$ and $j$ by a path following the arrows, then $i \to j$. The diagram is useful to determine whether a finite-state Markov chain is irreducible or not, or to check for periodicities. Draw the state transition diagrams and classify the states of the Markov chains with the following transition probability matrices:

(*a*) $$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

(*b*) $$P = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

(*c*) $$P = \begin{bmatrix} 0.3 & 0.4 & 0 & 0 & 0.3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

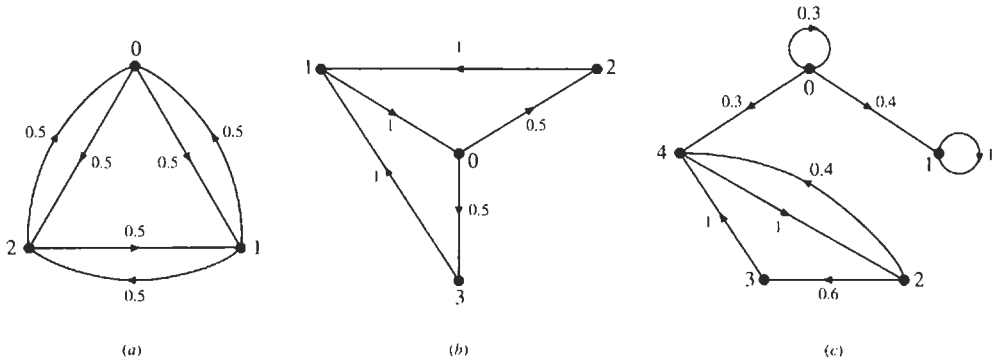



**Fig. 5-10** State transition diagram.

(*a*) The state transition diagram of the Markov chain with $P$ of part (*a*) is shown in Fig. 5-10(*a*). From Fig. 5-10(*a*), it is seen that the Markov chain is irreducible and aperiodic. For instance, one can get back to state 0 in two steps by going from 0 to 1 to 0. However, one can also get back to state 0 in three steps by going from 0 to 1 to 2 to 0. Hence 0 is aperiodic. Similarly, we can see that states 1 and 2 are also aperiodic.

(*b*) The state transition diagram of the Markov chain with $P$ of part (*b*) is shown in Fig. 5-10(*b*). From Fig. 5-10(*b*), it is seen that the Markov chain is irreducible and periodic with period 3.

(*c*) The state transition diagram of the Markov chain with $P$ of part (*c*) is shown in Fig. 5-10(*c*). From Fig. 5-10(*c*), it is seen that the Markov chain is not irreducible, since states 0 and 4 do not communicate, and state 1 is absorbing.

**5.36.** Consider a Markov chain with state space $\{0, 1\}$ and transition probability matrix

$$P = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(*a*) Show that state 0 is recurrent.

(*b*) Show that state 1 is transient.

(*a*) By Eqs. (*5.41*) and (*5.42*), we have

$$f_{00}^{(1)} = p_{00} = 1 \qquad f_{10}^{(1)} = p_{10} = \tfrac{1}{2}$$
$$f_{00}^{(2)} = p_{01}\, f_{10}^{(1)} = (0)\tfrac{1}{2} = 0$$
$$f_{00}^{(n)} = 0 \qquad n \geq 2$$

Then, by Eqs. (*5.43*),

$$f_{00} = P(T_0 < \infty \mid X_0 = 0) = \sum_{n=0}^{\infty} f_{00}^{(n)} = 1 + 0 + 0 + \cdots = 1$$

Thus, by definition (*5.44*), state 0 is recurrent.

(*b*) Similarly, we have

$$f_{11}^{(1)} = p_{11} = \tfrac{1}{2} \qquad f_{01}^{(1)} = p_{01} = 0$$
$$f_{11}^{(2)} = p_{10}\, f_{01}^{(1)} = (\tfrac{1}{2})0 = 0$$
$$f_{11}^{(n)} = 0 \qquad n \geq 2$$

and
$$f_{11} = P(T_1 < \infty \mid X_0 = 1) = \sum_{n=0}^{\infty} f_{11}^{(n)} = \tfrac{1}{2} + 0 + 0 + \cdots = \tfrac{1}{2} < 1$$

Thus, by definition (*5.48*), state 1 is transient.

**5.37.** Consider a Markov chain with state space $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Show that state 0 is periodic with period 2.

The characteristic equation of $P$ is given by

$$c(\lambda) = |\lambda I - P| = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{2} \\ -1 & \lambda & 0 \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^3 - \lambda = 0$$

Thus, by the Cayley-Hamilton theorem (in matrix analysis), we have $P^3 = P$. Thus, for $n \geq 1$,

$$P^{(2n)} = P^2 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$P^{(2n+1)} = P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Therefore $$d(0) = \gcd\{n \geq 1 : p_{00}^{(n)} > 0\} = \gcd\{2, 5, 6, \ldots\} = 2$$

Thus, state 0 is periodic with period 2.

Note that the state transition diagram corresponding to the given $P$ is shown in Fig. 5-11. From Fig. 5-11, it is clear that state 0 is periodic with period 2.

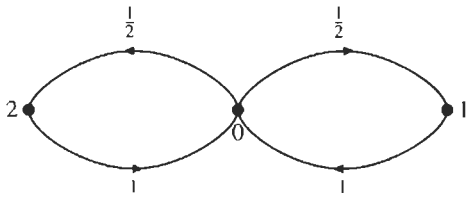



**Fig. 5-11**

**5.38.** Let two gamblers, A and B, initially have $k$ dollars and $m$ dollars, respectively. Suppose that at each round of their game, A wins one dollar from B with probability $p$ and loses one dollar to B with probability $q = 1 - p$. Assume that A and B play until one of them has no money left. (This is known as the *Gambler's Ruin* problem.) Let $X_n$ be A's capital after round $n$, where $n = 0, 1, 2, \ldots$ and $X_0 = k$.

(*a*) Show that $X(n) = \{X_n, n \geq 0\}$ is a Markov chain with absorbing states.

(*b*) Find its transition probability matrix $P$.

(*a*) The total capital of the two players at all times is

$$k + m = N$$

Let $Z_n$ $(n \geq 1)$ be independent r.v.'s with $P(Z_n = 1) = p$ and $P(Z_n = -1) = q = 1 - p$ for all $n$. Then

$$X_n = X_{n-1} + Z_n \qquad n = 1, 2, \ldots$$

and $X_0 = k$. The game ends when $X_n = 0$ or $X_n = N$. Thus, by Probs. 5.2 and 5.24, $X(n) = \{X_n, n \geq 0\}$ is a Markov chain with state space $E = \{0, 1, 2, \ldots, N\}$, where states 0 and $N$ are absorbing states. The Markov chain $X(n)$ is also known as a simple random walk with *absorbing barriers*.

(*b*) Since

$$\begin{aligned} p_{i,i+1} &= P(X_{n+1} = i + 1 \mid X_n = i) = p \\ p_{i,i-1} &= P(X_{n+1} = i - 1 \mid X_n = i) = q \\ p_{i,i} &= P(X_{n+1} = i \mid X_n = i) = 0 \qquad i \neq 0, N \\ p_{0,0} &= P(X_{n+1} = 0 \mid X_n = 0) = 1 \\ p_{N,N} &= P(X_{n+1} = N \mid X_n = N) = 1 \end{aligned}$$

the transition probability matrix $P$ is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ q & 0 & p & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q & 0 & p & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \ddots & & & \vdots \\ \vdots & & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \tag{5.123}$$

For example, when $p = q = \frac{1}{2}$ and $N = 4$,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

**5.39.** Consider a homogeneous Markov chain $X(n) = \{X_n, n \geq 0\}$ with a finite state space $E = \{0, 1, \ldots, N\}$, of which $A = \{0, 1, \ldots, m\}$, $m \geq 1$, is a set of absorbing states and $B = \{m + 1, \ldots, N\}$ is a set of nonabsorbing states. It is assumed that at least one of the absorbing states in $A$ is accessible from any nonabsorbing states in $B$. Show that absorption of $X(n)$ in one or another of the absorbing states is certain.

If $X_0 \in A$, then there is nothing to prove, since $X(n)$ is already absorbed. Let $X_0 \in B$. By assumption, there is at least one state in $A$ which is accessible from any state in $B$. Now assume that state $k \in A$ is accessible from $j \in B$. Let $n_{jk}$ $(< \infty)$ be the smallest number $n$ such that $p_{jk}^{(n)} > 0$. For a given state $j$, let $n_j$ be the largest of $n_{jk}$ as $k$ varies and $n'$ be the largest of $n_j$ as $j$ varies. After $n'$ steps, no matter what the initial state of $X(n)$, there is a probability $p > 0$ that $X(n)$ is in an absorbing state. Therefore

$$P\{X_{n'} \in B\} = 1 - p$$

and $0 < 1 - p < 1$. It follows by homogeneity and the Markov property that

$$P\{X_{k(n')} \in B\} = (1 - p)^k \qquad k = 1, 2, \ldots$$

Now since $\lim_{k\to\infty}(1-p)^k = 0$, we have

$$\lim_{n\to\infty} P\{X_n \in B\} = 0 \qquad \text{or} \qquad \lim_{n\to\infty} P\{X_n \in \bar{B} = A\} = 1$$

which shows that absorption of $X(n)$ in one or another of the absorption states is certain.

**5.40.** Verify Eq. (*5.50*).

Let $X(n) = \{X_n, n \geq 0\}$ be a homogeneous Markov chain with a finite state space $E = \{0, 1, \ldots, N\}$, of which $A = \{0, 1, \ldots, m\}$, $m \geq 1$, is a set of absorbing states and $B = \{m+1, \ldots, N\}$ is a set of nonabsorbing states. Let state $k \in B$ at the first step go to $i \in E$ with probability $p_{ki}$. Then

$$\begin{aligned} u_{kj} &= P\{X_n = j(\in A) \mid X_0 = k(\in B)\} \\ &= \sum_{i=1}^{N} p_{ki} P\{X_n = j(\in A) \mid X_0 = i\} \end{aligned} \tag{5.124}$$

Now

$$P\{X_n = j(\in A),\ X_0 = i\} = \begin{cases} 1 & i = j \\ 0 & i \in A,\ i \neq j \\ u_{ij} & i \in B,\ i = m+1, \ldots, N \end{cases}$$

Then Eq. (*5.124*) becomes

$$u_{kj} = p_{kj} + \sum_{i=m+1}^{N} p_{ki} u_{ij} \qquad k = m+1, \ldots, N;\ j = 1, \ldots, m \tag{5.125}$$

But $p_{kj}$, $k = m+1, \ldots, N$; $j = 1, \ldots, m$, are the elements of $R$, whereas $p_{ki}$, $k = m+1, \ldots, N$; $i = m+1, \ldots, N$ are the elements of $Q$ [see Eq. (*5.49a*)]. Hence, in matrix notation, Eq. (*5.125*) can be expressed as

$$U = R + QU \qquad \text{or} \qquad (I - Q)U = R \tag{5.126}$$

Premultiplying both sides of the second equation of Eq. (*5.126*) with $(I-Q)^{-1}$, we obtain

$$U = (I - Q)^{-1}R = \Phi R$$

**5.41.** Consider a simple random walk $X(n)$ with absorbing barriers at state 0 and state $N = 3$ (see Prob. 5.38).

(*a*) Find the transition probability matrix $P$.

(*b*) Find the probabilities of absorption into states 0 and 3.

(*a*) The transition probability matrix $P$ is [Eq. (*5.123*)]

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(*b*) Rearranging the transition probability matrix $P$ as [Eq. (*5.49a*)],

$$P = \begin{array}{c} \\ 0 \\ 3 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 3 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & p & q & 0 \end{bmatrix} \end{array}$$

and by Eq. (*5.49b*), the matrices $Q$ and $R$ are given by

$$R = \begin{bmatrix} p_{10} & p_{13} \\ p_{20} & p_{23} \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} \qquad Q = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

Then

$$I - Q = \begin{bmatrix} 1 & -p \\ -q & 1 \end{bmatrix}$$

and

$$\Phi = (I - Q)^{-1} = \frac{1}{1 - pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix} \tag{5.127}$$

By Eq. (*5.50*),

$$U = \begin{bmatrix} u_{10} & u_{13} \\ u_{20} & u_{23} \end{bmatrix} = \Phi R = \frac{1}{1 - pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix} = \frac{1}{1 - pq} \begin{bmatrix} q & p^2 \\ q^2 & p \end{bmatrix} \tag{5.128}$$

Thus, the probabilities of absorption into state 0 from states 1 and 2 are given, respectively, by

$$u_{10} = \frac{q}{1 - pq} \qquad \text{and} \qquad u_{20} = \frac{q^2}{1 - pq}$$

and the probabilities of absorption into state 3 from states 1 and 2 are given, respectively, by

$$u_{13} = \frac{p^2}{1 - pq} \qquad \text{and} \qquad u_{23} = \frac{p}{1 - pq}$$

Note that

$$u_{10} + u_{13} = \frac{q + p^2}{1 - pq} = \frac{1 - p + p^2}{1 - p(1 - p)} = 1$$

$$u_{20} + u_{23} = \frac{q^2 + p}{1 - pq} = \frac{q^2 + (1 - q)}{1 - (1 - q)q} = 1$$

which confirm the proposition of Prob. 5.39.

**5.42.** Consider the simple random walk $X(n)$ with absorbing barriers at 0 and 3 (Prob. 5.41). Find the expected time (or steps) to absorption when $X_0 = 1$ and when $X_0 = 2$.

The fundamental matrix $\Phi$ of $X(n)$ is [Eq. (*5.127*)]

$$\Phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \frac{1}{1 - pq} \begin{bmatrix} 1 & p \\ q & 1 \end{bmatrix}$$

Let $T_i$ be the time to absorption when $X_0 = i$. Then by Eq. (*5.51*), we get

$$E(T_1) = \frac{1}{1 - pq}(1 + p) \qquad E(T_2) = \frac{1}{1 - pq}(q + 1) \tag{5.129}$$

**5.43.** Consider the gambler's game described in Prob. 5.38. What is the probability of A's losing all his money?

Let $P(k)$, $k = 0, 1, 2, \ldots, N$, denote the probability that A loses all his money when his initial capital is $k$ dollars. Equivalently, $P(k)$ is the probability of absorption at state 0 when $X_0 = k$ in the simple random

walk $X(n)$ with absorbing barriers at states 0 and $N$. Now if $0 < k < N$, then

$$P(k) = pP(k+1) + qP(k-1) \qquad k = 1, 2, \ldots, N-1 \tag{5.130}$$

where $pP(k+1)$ is the probability that A wins the first round and subsequently loses all his money and $qP(k-1)$ is the probability that A loses the first round and subsequently loses all his money. Rewriting Eq. (5.130), we have

$$P(k+1) - \frac{1}{p}P(k) + \frac{q}{p}P(k-1) = 0 \qquad k = 1, 2, \ldots, N-1 \tag{5.131}$$

which is a second-order homogeneous linear constant-coefficient difference equation. Next, we have

$$P(0) = 1 \qquad \text{and} \qquad P(N) = 0 \tag{5.132}$$

since if $k = 0$, absorption at 0 is a sure event, and if $k = N$, absorption at $N$ has occurred and absorption at 0 is impossible. Thus, finding $P(k)$ reduces to solving Eq. (5.131) subject to the boundary conditions given by Eq. (5.132). Let $P(k) = r^k$. Then Eq. (5.131) becomes

$$r^{k+1} - \frac{1}{p}r^k + \frac{q}{p}r^{k-1} = 0 \qquad p + q = 1$$

Setting $k = 1$ (and noting that $p + q = 1$), we get

$$r^2 - \frac{1}{p}r + \frac{q}{p} = (r-1)\left(r - \frac{q}{p}\right) = 0$$

from which we get $r = 1$ and $r = q/p$. Thus,

$$P(k) = c_1 + c_2\left(\frac{q}{p}\right)^k \qquad q \neq p \tag{5.133}$$

where $c_1$ and $c_2$ are arbitrary constants. Now, by Eq. (5.132),

$$P(0) = 1 \rightarrow c_1 + c_2 = 1$$
$$P(N) = 0 \rightarrow c_1 + c_2\left(\frac{q}{p}\right)^N = 0$$

Solving for $c_1$ and $c_2$, we obtain

$$c_1 = \frac{-(q/p)^N}{1-(q/p)^N} \qquad c_2 = \frac{1}{1-(q/p)^N}$$

Hence

$$P(k) = \frac{(q/p)^k - (q/p)^N}{1-(q/p)^N} \qquad q \neq p \tag{5.134}$$

Note that if $N \gg k$,

$$P(k) = \begin{cases} 1 & q > p \\ \left(\dfrac{q}{p}\right)^k & p > q \end{cases} \tag{5.135}$$

Setting $r = q/p$ in Eq. (5.134), we have

$$P(k) = \frac{r^k - r^N}{1-r^N} \xrightarrow[r \to 1]{} 1 - \frac{k}{N}$$

Thus, when $p = q = \frac{1}{2}$,

$$P(k) = 1 - \frac{k}{N} \tag{5.136}$$

**5.44.** Show that Eq. (5.134) is consistent with Eq. (5.128).

Substituting $k = 1$ and $N = 3$ in Eq. (*5.134*), and noting that $p + q = 1$, we have

$$P(1) = \frac{(q/p) - (q/p)^3}{1 - (q/p)^3} = \frac{q(p^2 - q^2)}{(p^3 - q^3)}$$
$$= \frac{q(p+q)}{p^2 + pq + q^2} = \frac{q}{(p+q)^2 - pq} = \frac{q}{1 - pq}$$

Now from Eq. (*5.128*), we have

$$u_{10} = \frac{q}{1 - pq} = P(1)$$

**5.45.** Consider the simple random walk $X(n)$ with state space $E = \{0, 1, 2, \ldots, N\}$, where 0 and $N$ are absorbing states (Prob. 5.38). Let r.v. $T_k$ denote the time (or number of steps) to absorption of $X(n)$ when $X_0 = k$, $k = 0, 1, \ldots, N$. Find $E(T_k)$.

Let $Y(k) = E(T_k)$. Clearly, if $k = 0$ or $k = N$, then absorption is immediate, and we have

$$Y(0) = Y(N) = 0 \tag{5.137}$$

Let the probability that absorption takes $m$ steps when $X_0 = k$ be defined by

$$P(k, m) = P(T_k = m) \qquad m = 1, 2, \ldots \tag{5.138}$$

Then, we have (Fig. 5-12)

$$P(k, m) = pP(k+1, m-1) + qP(k-1, m-1) \tag{5.139}$$

and
$$Y(k) = E(T_k) = \sum_{m=1}^{\infty} mP(k, m) = p\sum_{m=1}^{\infty} mP(k+1, m-1) + q\sum_{m=1}^{\infty} mP(k-1, m-1)$$

Setting $m - 1 = i$, we get

$$Y(k) = p\sum_{i=0}^{\infty}(i+1)P(k+1, i) + q\sum_{i=0}^{\infty}(i+1)P(k-1, i)$$
$$= p\sum_{i=0}^{\infty} iP(k+1, i) + q\sum_{i=0}^{\infty} iP(k-1, i) + p\sum_{i=0}^{\infty} P(k+1, i) + q\sum_{i=0}^{\infty} P(k-1, i)$$

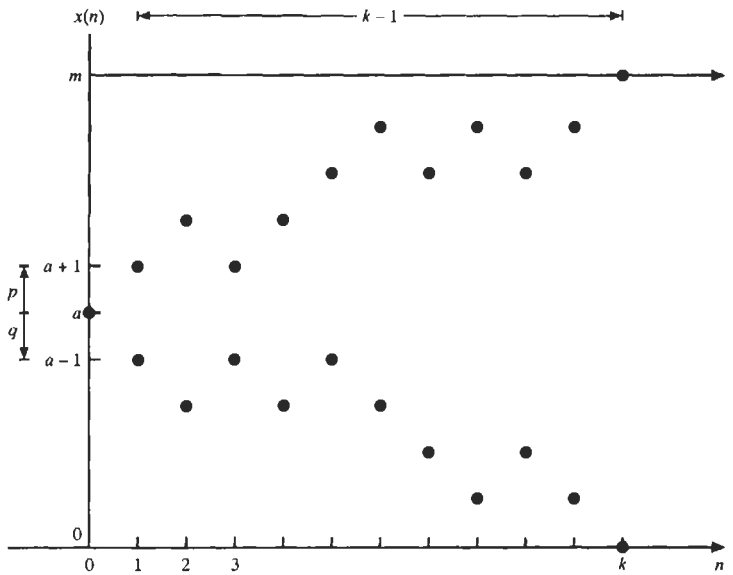



**Fig. 5-12** Simple random walk with absorbing barriers.

Now by the result of Prob. 5.39, we see that absorption is certain; therefore

$$\sum_{i=0}^{\infty} P(k+1, i) = \sum_{i=0}^{\infty} P(k-1, i) = 1$$

Thus $$Y(k) = pY(k+1) + qY(k-1) + p + q$$

or $$Y(k) = pY(k+1) + qY(k-1) + 1 \qquad k = 1, 2, \ldots, N-1 \tag{5.140}$$

Rewriting Eq. (*5.140*), we have

$$Y(k+1) - \frac{1}{p} Y(k) + \frac{q}{p} Y(k-1) = -\frac{1}{p} \tag{5.141}$$

Thus, finding $P(k)$ reduces to solving Eq. (*5.141*) subject to the boundary conditions given by Eq. (*5.137*). Let the general solution of Eq. (*5.141*) be

$$Y(k) = Y_h(k) + Y_p(k)$$

where $Y_h(k)$ is the homogeneous solution satisfying

$$Y_h(k+1) - \frac{1}{p} Y_h(k) + \frac{q}{p} Y_h(k-1) = 0 \tag{5.142}$$

and $Y_p(k)$ is the particular solution satisfying

$$Y_p(k+1) - \frac{1}{p} Y_p(k) + \frac{q}{p} Y_p(k-1) = -\frac{1}{p} \tag{5.143}$$

Let $Y_p(k) = \alpha k$, where $\alpha$ is a constant. Then Eq. (*5.143*) becomes

$$(k+1)\alpha - \frac{1}{p} k\alpha + \frac{q}{p} (k-1)\alpha = -\frac{1}{p}$$

from which we get $\alpha = 1/(q-p)$ and

$$Y_p(k) = \frac{k}{q-p} \qquad p \neq q \tag{5.144}$$

Since Eq. (*5.142*) is the same as Eq. (*5.131*), by Eq. (*5.133*), we obtain

$$Y_h(k) = c_1 + c_2\left(\frac{q}{p}\right)^k \qquad q \neq p \tag{5.145}$$

where $c_1$ and $c_2$ are arbitrary constants. Hence, the general solution of Eq. (*5.141*) is

$$Y(k) = c_1 + c_2\left(\frac{q}{p}\right)^k + \frac{k}{q-p} \qquad q \neq p \tag{5.146}$$

Now, by Eq. (*5.137*),

$$Y(0) = 0 \rightarrow c_1 + c_2 = 0$$
$$Y(N) = 0 \rightarrow c_1 + c_2\left(\frac{q}{p}\right)^N + \frac{N}{q-p} = 0$$

Solving for $c_1$ and $c_2$, we obtain

$$c_1 = \frac{-N/(q-p)}{1-(q/p)^N} \qquad c_2 = \frac{N/(q-p)}{1-(q/p)^N}$$

Substituting these values in Eq. (*5.146*), we obtain (for $p \neq q$)

$$Y(k) = E(T_k) = \frac{1}{q-p}\left(k - N\left[\frac{1-(q/p)^k}{1-(q/p)^N}\right]\right) \tag{5.147}$$

When $p = q = \frac{1}{2}$, we have

$$Y(k) = E(T_k) = k(N-k) \qquad p = q = \tfrac{1}{2} \tag{5.148}$$

**5.46.** Consider a Markov chain with two states and transition probability matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(*a*) Find the stationary distribution $\hat{\mathbf{p}}$ of the chain.

(*b*) Find $\lim_{n\to\infty} P^n$.

(*a*) By definition (*5.52*),

$$\hat{\mathbf{p}}P = \hat{\mathbf{p}}$$

or

$$[p_1 \quad p_2]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [p_1 \quad p_2]$$

which yields $p_1 = p_2$. Since $p_1 + p_2 = 1$, we obtain

$$\hat{\mathbf{p}} = [\tfrac{1}{2} \quad \tfrac{1}{2}]$$

(*b*) Now

$$P^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & n = 1, 3, 5, \ldots \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & n = 2, 4, 6, \ldots \end{cases}$$

and $\lim_{n\to\infty} P^n$ does not exist.

**5.47.** Consider a Markov chain with two states and transition probability matrix

$$P = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

(*a*) Find the stationary distribution $\hat{\mathbf{p}}$ of the chain.

(*b*) Find $\lim_{n\to\infty} P^n$.

(*c*) Find $\lim_{n\to\infty} P^n$ by first evaluating $P^n$.

(*a*) By definition (*5.52*), we have

$$\hat{\mathbf{p}}P = \hat{\mathbf{p}}$$

or

$$[p_1 \quad p_2]\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = [p_1 \quad p_2]$$

which yields

$$\tfrac{3}{4}p_1 + \tfrac{1}{2}p_2 = p_1$$
$$\tfrac{1}{4}p_1 + \tfrac{1}{2}p_2 = p_2$$

Each of these equations is equivalent to $p_1 = 2p_2$. Since $p_1 + p_2 = 1$, we obtain

$$\hat{\mathbf{p}} = [\tfrac{2}{3} \quad \tfrac{1}{3}]$$

(*b*) Since the Markov chain is regular, by Eq. (*5.53*), we obtain

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty}\begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^n = \begin{bmatrix} \hat{\mathbf{p}} \\ \hat{\mathbf{p}} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

(*c*) Setting $a = \frac{1}{4}$ and $b = \frac{1}{2}$ in Eq. (*5.120*) (Prob. 5.30), we get

$$P^n = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix} - (\tfrac{1}{4})^n \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Since $\lim_{n\to\infty} (\frac{1}{4})^n = 0$, we obtain

$$\lim_{n\to\infty} P^n = \lim_{n\to\infty} \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}^n = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

## POISSON PROCESSES

**5.48.** Let $T_n$ denote the arrival time of the $n$th customer at a service station. Let $Z_n$ denote the time interval between the arrival of the $n$th customer and the $(n-1)$st customer; that is,

$$Z_n = T_n - T_{n-1} \qquad n \geq 1 \tag{5.149}$$

and $T_0 = 0$. Let $\{X(t), t \geq 0\}$ be the counting process associated with $\{T_n, n \geq 0\}$. Show that if $X(t)$ has stationary increments, then $Z_n$, $n = 1, 2, \ldots$, are identically distributed r.v.'s.

We have

$$P(Z_n > z) = 1 - P(Z_n \leq z) = 1 - F_{Z_n}(z)$$

By Eq. (*5.149*), $$P(Z_n > z) = P(T_n - T_{n-1} > z) = P(T_n > T_{n-1} + z)$$

Suppose that the observed value of $T_{n-1}$ is $t_{n-1}$. The event $(T_n > T_{n-1} + z \mid T_{n-1} = t_{n-1})$ occurs if and only if $X(t)$ does not change count during the time interval $(t_{n-1}, t_{n-1} + z)$ (Fig. 5-13). Thus,

$$\begin{aligned} P(Z_n > z \mid T_{n-1} = t_{n-1}) &= P(T_n > T_{n-1} + z \mid T_{n-1} = t_{n-1}) \\ &= P[X(t_{n-1} + z) - X(t_{n-1}) = 0] \end{aligned}$$

or $$P(Z_n \leq z \mid T_{n-1} = t_{n-1}) = 1 - P[X(t_{n-1} + z) - X(t_{n-1}) = 0] \tag{5.150}$$

Since $X(t)$ has stationary increments, the probability on the right-hand side of Eq. (*5.150*) is a function only of the time difference $z$. Thus

$$P(Z_n \leq z \mid T_{n-1} = t_{n-1}) = 1 - P[X(z) = 0] \tag{5.151}$$

which shows that the conditional distribution function on the left-hand side of Eq. (*5.151*) is independent of the particular value of $n$ in this case, and hence we have

$$F_{Z_n}(z) = P(Z_n \leq z) = 1 - P[X(z) = 0] \tag{5.152}$$

which shows that the cdf of $Z_n$ is independent of $n$. Thus we conclude that the $Z_n$'s are identically distributed r.v.'s.

**Fig. 5-13**

**5.49.** Show that Definition 5.6.2 implies Definition 5.6.1.

Let $p_n(t) = P[X(t) = n]$. Then, by condition 2 of Definition 5.6.2, we have

$$\begin{aligned} p_0(t + \Delta t) &= P[X(t + \Delta t) = 0] = P[X(t) = 0, X(t + \Delta t) - X(0) = 0] \\ &= P[X(t) = 0]P[X(t + \Delta t) - X(t) = 0] \end{aligned}$$

Now, by Eq. (5.59), we have

$$P[X(t + \Delta t) - X(t) = 0] = 1 - \lambda\, \Delta t + o(\Delta t)$$

Thus,
$$p_0(t + \Delta t) = p_0(t)[1 - \lambda\, \Delta t + o(\Delta t)]$$

or
$$\frac{p_0(t + \Delta t) - p_0(t)}{\Delta t} = -\lambda p_0(t) + \frac{o(\Delta t)}{\Delta t}$$

Letting $\Delta t \to 0$, and by Eq. (5.58), we obtain

$$p_0'(t) = -\lambda p_0(t) \tag{5.153}$$

Solving the above differential equation, we get

$$p_0(t) = ke^{-\lambda t}$$

where $k$ is an integration constant. Since $p_0(0) = P[X(0) = 0] = 1$, we obtain

$$p_0(t) = e^{-\lambda t} \tag{5.154}$$

Similarly, for $n > 0$,

$$\begin{aligned} p_n(t + \Delta t) &= P[X(t + \Delta t) = n] \\ &= P[X(t) = n,\ X(t + \Delta t) - X(0) = 0] \\ &\quad + P[X(t) = n - 1,\ X(t + \Delta t) - X(0) = 1] + \sum_{k=2}^{n} P[X(t) = n - k,\ X(t + \Delta t) - X(0) = k] \end{aligned}$$

Now, by condition 4 of Definition 5.6.2, the last term in the above expression is $o(\Delta t)$. Thus, by conditions 2 and 3 of Definition 5.6.2, we have

$$p_n(t + \Delta t) = p_n(t)[1 - \lambda\, \Delta t + o(\Delta t)] + p_{n-1}(t)[\lambda\, \Delta t + o(\Delta t)] + o(\Delta t)$$

Thus
$$\frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{o(\Delta t)}{\Delta t}$$

and letting $\Delta t \to 0$ yields

$$p_n'(t) + \lambda p_n(t) = \lambda p_{n-1}(t) \tag{5.155}$$

Multiplying both sides by $e^{\lambda t}$, we get

$$e^{\lambda t}[p_n'(t) + \lambda p_n(t)] = \lambda e^{\lambda t} p_{n-1}(t)$$

Hence
$$\frac{d}{dt}\,[e^{\lambda t} p_n(t)] = \lambda e^{\lambda t} p_{n-1}(t) \tag{5.156}$$

Then by Eq. (5.154), we have

$$\frac{d}{dt}\,[e^{\lambda t} p_1(t)] = \lambda$$

or
$$p_1(t) = (\lambda t + c)e^{-\lambda t}$$

where $c$ is an integration constant. Since $p_1(0) = P[X(0) = 1] = 0$, we obtain

$$p_1(t) = \lambda t e^{-\lambda t} \tag{5.157}$$

To show that

$$p_n(t) = e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}$$

we use mathematical induction. Assume that it is true for $n - 1$; that is,

$$p_{n-1}(t) = e^{-\lambda t}\,\frac{(\lambda t)^{n-1}}{(n-1)!}$$

Substituting the above expression into Eq. (5.156), we have

$$\frac{d}{dt}[e^{\lambda t}p_n(t)] = \frac{\lambda^n t^{n-1}}{(n-1)!}$$

Integrating, we get

$$e^{\lambda t}p_n(t) = \frac{(\lambda t)^n}{n!} + c_1$$

Since $p_n(0) = 0$, $c_1 = 0$, and we obtain

$$p_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{5.158}$$

which is Eq. (5.55) of Definition 5.6.1. Thus we conclude that Definition 5.6.2 implies Definition 5.6.1.

**5.50.** Verify Eq. (5.59).

We note first that $X(t)$ can assume only nonnegative integer values; therefore, the same is true for the counting increment $X(t + \Delta t) - X(t)$. Thus, summing over all possible values of the increment, we get

$$\begin{aligned}\sum_{k=0}^{\infty} P[X(t+\Delta t) - X(t) = k] &= P[X(t+\Delta t) - X(t) = 0] \\ &\quad + P[X(t+\Delta t) - X(t) = 1] + P[X(t+\Delta t) - X(t) \geq 2] \\ &= 1\end{aligned}$$

Substituting conditions 3 and 4 of Definition 5.6.2 into the above equation, we obtain

$$P[X(t+\Delta t) - X(t) = 0] = 1 - \lambda\,\Delta t + o(\Delta t)$$

**5.51.** (*a*) Using the Poison probability distribution in Eq. (5.158), obtain an analytical expression for the correction term $o(\Delta t)$ in the expression (condition 3 of Definition 5.6.2)

$$P[X(t+\Delta t) - X(t) = 1] = \lambda\,\Delta t + o(\Delta t) \tag{5.159}$$

(*b*) Show that this correction term does have the property of Eq. (5.58); that is,

$$\lim_{\Delta t \to 0}\frac{o(\Delta t)}{\Delta t} = 0$$

(*a*) Since the Poisson process $X(t)$ has stationary increments, Eq. (5.159) can be rewritten as

$$P[X(\Delta t) = 1] = p_1(\Delta t) = \lambda\,\Delta t + o(\Delta t) \tag{5.160}$$

Using Eq. (5.158) [or Eq. (5.157)], we have

$$\begin{aligned}p_1(\Delta t) &= \lambda\,\Delta t\, e^{-\lambda\,\Delta t} = \lambda\,\Delta t(1 + e^{-\lambda\,\Delta t} - 1) \\ &= \lambda\,\Delta t + \lambda\,\Delta t(e^{-\lambda\,\Delta t} - 1)\end{aligned}$$

Equating the above expression with Eq. (5.160), we get

$$\lambda\,\Delta t + o(\Delta t) = \lambda\,\Delta t + \lambda\,\Delta t(e^{-\lambda\,\Delta t} - 1)$$

from which we obtain

$$o(\Delta t) = \lambda\,\Delta t(e^{-\lambda\,\Delta t} - 1) \tag{5.161}$$

(*b*) From Eq. (5.161), we have

$$\lim_{\Delta t \to 0}\frac{o(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0}\frac{\lambda\,\Delta t(e^{-\lambda\,\Delta t} - 1)}{\Delta t} = \lim_{\Delta t \to 0}\lambda(e^{-\lambda\,\Delta t} - 1) = 0$$

**5.52.** Find the autocorrelation function $R_X(t, s)$ and the autocovariance function $K_X(t, s)$ of a Poisson process $X(t)$ with rate $\lambda$.

From Eqs. (5.56) and (5.57),

$$E[X(t)] = \lambda t \qquad \text{Var}[X(t)] = \lambda t$$

Now, the Poisson process $X(t)$ is a random process with stationary independent increments and $X(0) = 0$. Thus, by Eq. (5.103) (Prob. 5.23), we obtain

$$K_X(t, s) = \sigma_1^2 \min(t, s) = \lambda \min(t, s) \tag{5.162}$$

since $\sigma_1^2 = \text{Var}[X(1)] = \lambda$. Next, since $E[X(t)]E[X(s)] = \lambda^2 ts$, by Eq. (5.10), we obtain

$$R_X(t, s) = \lambda \min(t, s) + \lambda^2 ts \tag{5.163}$$

**5.53.** Show that the time intervals between successive events (or interarrival times) in a Poisson process $X(t)$ with rate $\lambda$ are independent and identically distributed exponential r.v.'s with parameter $\lambda$.

Let $Z_1, Z_2, \ldots$ be the r.v.'s representing the lengths of interarrival times in the Poisson process $X(t)$. First, notice that $\{Z_1 > t\}$ takes place if and only if no event of the Poisson process occur in the interval $(0, t)$, and thus by Eq. (5.154),

$$P(Z_1 > t) = P\{X(t) = 0\} = e^{-\lambda t}$$

or

$$F_{Z_1}(t) = P(Z_1 \le t) = 1 - e^{-\lambda t}$$

Hence $Z_1$ is an exponential r.v. with parameter $\lambda$ [Eq. (2.49)]. Let $f_1(t)$ be the pdf of $Z_1$. Then we have

$$\begin{aligned} P(Z_2 > t) &= \int P(Z_2 > t \mid Z_1 = \tau) f_1(\tau)\, d\tau \\ &= \int P[X(t + \tau) - X(\tau) = 0] f_1(\tau)\, d\tau \\ &= e^{-\lambda t} \int f_1(\tau)\, d\tau = e^{-\lambda t} \end{aligned} \tag{5.164}$$

which indicates that $Z_2$ is also an exponential r.v. with parameter $\lambda$ and is independent of $Z_1$. Repeating the same argument, we conclude that $Z_1, Z_2, \ldots$ are iid exponential r.v.'s with parameter $\lambda$.

**5.54.** Let $T_n$ denote the time of the $n$th event of a Poisson process $X(t)$ with rate $\lambda$. Show that $T_n$ is a gamma r.v. with parameters $(n, \lambda)$.

Clearly,

$$T_n = Z_1 + Z_2 + \cdots + Z_n$$

where $Z_n$, $n = 1, 2, \ldots$, are the interarrival times defined by Eq. (5.149). From Prob. 5.53, we know that $Z_n$ are iid exponential r.v.'s with parameter $\lambda$. Now, using the result of Prob. 4.33, we see that $T_n$ is a gamma r.v. with parameters $(n, \lambda)$, and its pdf is given by [Eq. (2.76)]:

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \dfrac{(\lambda t)^{n-1}}{(n-1)!} & t > 0 \\ 0 & t < 0 \end{cases} \tag{5.165}$$

The random process $\{T_n, n \ge 1\}$ is often called an *arrival process*.

**5.55.** Suppose $t$ is not a point at which an event occurs in a Poisson process $X(t)$ with rate $\lambda$. Let $W(t)$ be the r.v. representing the time until the next occurrence of an event. Show that the distribution of $W(t)$ is independent of $t$ and $W(t)$ is an exponential r.v. with parameter $\lambda$.

Let $s$ $(0 \le s < t)$ be the point at which the last event [say the $(n-1)$st event] occurred (Fig. 5-14). The event $\{W(t) > \tau\}$ is equivalent to the event

$$\{Z_n > t - s + \tau \mid Z_n > t - s\}$$

**Fig. 5-14**

Thus, using Eq. (*5.164*), we have

$$P[W(t) > \tau] = P(Z_n > t - s + \tau \mid Z_n > t - s) = \frac{P(Z_n > t - s + \tau)}{P(Z_n > t - s)} = \frac{e^{-\lambda(t-s+\tau)}}{e^{-\lambda(t-s)}} = e^{-\lambda\tau}$$

and

$$P[W(t) \le \tau] = 1 - e^{\lambda\tau} \tag{5.166}$$

which indicates that $W(t)$ is an exponential r.v. with parameter $\lambda$ and is independent of $t$. Note that $W(t)$ is often called a *waiting time*.

**5.56.** Patients arrive at the doctor's office according to a Poisson process with rate $\lambda = \frac{1}{10}$ minute. The doctor will not see a patient until at least three patients are in the waiting room.

(*a*) Find the expected waiting time until the first patient is admitted to see the doctor.

(*b*) What is the probability that nobody is admitted to see the doctor in the first hour?

(*a*) Let $T_n$ denote the arrival time of the $n$th patient at the doctor's office. Then

$$T_n = Z_1 + Z_2 + \cdots + Z_n$$

where $Z_n$, $n = 1, 2, \ldots$, are iid exponential r.v.'s with parameter $\lambda = \frac{1}{10}$. By Eqs. (*4.108*) and (*2.50*),

$$E(T_n) = E\left(\sum_{i=1}^{n} Z_i\right) = \sum_{i=i}^{n} E(Z_i) = n\,\frac{1}{\lambda} \tag{5.167}$$

The expected waiting time until the first patient is admitted to see the doctor is

$$E(T_3) = 3(10) = 30 \text{ minutes}$$

(*b*) Let $X(t)$ be the Poisson process with parameter $\lambda = \frac{1}{10}$. The probability that nobody is admitted to see the doctor in the first hour is the same as the probability that at most two patients arrive in the first 60 minutes. Thus, by Eq. (*5.55*),

$$\begin{aligned} P[X(60) - X(0) \le 2] &= P[X(60) - X(0) = 0] + P[X(60) - X(0) = 1] + P[X(60) - X(0) = 2] \\ &= e^{-60/10} + e^{-60/10}\left(\tfrac{60}{10}\right) + e^{-60/10}\tfrac{1}{2}\left(\tfrac{60}{10}\right)^2 \\ &= e^{-6}(1 + 6 + 18) \approx 0.062 \end{aligned}$$

**5.57.** Let $T_n$ denote the time of the $n$th event of a Poisson process $X(t)$ with rate $\lambda$. Suppose that one event has occurred in the interval $(0, t)$. Show that the conditional distribution of arrival time $T_1$ is uniform over $(0, t)$.

For $\tau \le t$,

$$\begin{aligned} P[T_1 \le \tau \mid X(t) = 1] &= \frac{P[T_1 \le \tau,\ X(t) = 1]}{P[X(t) = 1]} \\ &= \frac{P[X(\tau) = 1,\ X(t) - X(\tau) = 0]}{P[X(t) = 1]} \\ &= \frac{P[X(\tau) = 1]P[X(t) - X(\tau) = 0]}{P[X(t) = 1]} \\ &= \frac{\lambda\tau e^{-\lambda\tau} e^{-\lambda(t-\tau)}}{\lambda t e^{-\lambda t}} = \frac{\tau}{t} \end{aligned} \tag{5.168}$$

which indicates that $T_1$ is uniform over $(0, t)$ [see Eq. (*2.45*)].

**5.58.** Consider a Poisson process $X(t)$ with rate $\lambda$, and suppose that each time an event occurs, it is classified as either a type 1 or a type 2 event. Suppose further that the event is classified as a type 1 event with probability $p$ and a type 2 event with probability $1 - p$. Let $X_1(t)$ and $X_2(t)$ denote the number of type 1 and type 2 events, respectively, occurring in $(0, t)$. Show that $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ are both Poisson processes with rates $\lambda p$ and $\lambda(1 - p)$, respectively. Furthermore, the two processes are independent.

We have

$$X(t) = X_1(t) + X_2(t)$$

First we calculate the joint probability $P[X_1(t) = k, X_2(t) = m]$.

$$P[X_1(t) = k, X_2(t) = m] = \sum_{n=0}^{\infty} P[X_1(t) = k, X_2(t) = m \mid X(t) = n]P[X(t) = n]$$

Note that

$$P[X_1(t) = k, X_2(t) = m \mid X(t) = n] = 0 \qquad \text{when } n \neq k + m$$

Thus, using Eq. (*5.158*), we obtain

$$\begin{aligned} P[X_1(t) = k, X_2(t) = m] &= P[X_1(t) = k, X_2(t) = m \mid X(t) = k + m]P[X(t) = k + m] \\ &= P[X_1(t) = k, X_2(t) = m \mid X(t) = k + m]e^{-\lambda t}\frac{(\lambda t)^{k+m}}{(k+m)!} \end{aligned}$$

Now, given that $k + m$ events occurred, since each event has probability $p$ of being a type 1 event and probability $1 - p$ of being a type 2 event, it follows that

$$P[X_1(t) = k, X_2(t) = m \mid X(t) = k + m] = \binom{k+m}{k} p^k(1-p)^m$$

Thus,

$$\begin{aligned} P[X_1(t) = k, X_2(t) = m] &= \binom{k+m}{k} p^k(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{k+m}}{(k+m)!} \\ &= \frac{(k+m)!}{k!\,m!}\, p^k(1-p)^m e^{-\lambda t}\frac{(\lambda t)^{k+m}}{(k+m)!} \\ &= e^{-\lambda pt}\frac{(\lambda pt)^k}{k!}\, e^{-\lambda(1-p)t}\frac{[\lambda(1-p)t]^m}{m!} \end{aligned} \tag{5.169}$$

Then

$$\begin{aligned} P[X_1(t) = k] &= \sum_{m=1}^{\infty} P[X_1(t) = k, X_2(t) = m] \\ &= e^{-\lambda pt}\frac{(\lambda pt)^k}{k!}\, e^{-\lambda(1-p)t}\sum_{m=1}^{\infty}\frac{[\lambda(1-p)t]^m}{m!} \\ &= e^{-\lambda pt}\frac{(\lambda pt)^k}{k!}\, e^{-\lambda(1-p)t}e^{\lambda(1-p)t} \\ &= e^{-\lambda pt}\frac{(\lambda pt)^k}{k!} \end{aligned} \tag{5.170}$$

which indicates that $X_1(t)$ is a Poisson process with rate $\lambda p$. Similarly, we can obtain

$$\begin{aligned} P[X_2(t) = m] &= \sum_{k=1}^{\infty} P[X_1(t) = k, X_2(t) = m] \\ &= e^{-\lambda(1-p)t}\frac{[\lambda(1-p)t]^m}{m!} \end{aligned} \tag{5.171}$$

and so $X_2(t)$ is a Poisson process with rate $\lambda(1 - p)$. Finally, from Eqs. (*5.170*), (*5.171*), and (*5.169*), we see that

$$P[X_1(t) = k, X_2(t) = m] = P[X_1(t) = k]P[X_2(t) = m]$$

Hence, $X_1(t)$ and $X_2(t)$ are independent.

## WIENER PROCESSES

**5.59.** Let $X_1, \ldots, X_n$ be jointly normal r.v.'s. Show that the joint characteristic function of $X_1, \ldots, X_n$ is given by

$$\Psi_{X_1 \cdots X_n}(\omega_1, \ldots, \omega_n) = \exp\left( j \sum_{i=1}^{n} \omega_i \mu_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_i \omega_k \sigma_{ik} \right) \tag{5.172}$$

where $\mu_i = E(X_i)$ and $\sigma_{ik} = \mathrm{Cov}(X_i, X_k)$.

Let $$Y = a_1 X_1 + a_2 X_2 + \cdots + a_n X_n$$

By definition (*4.50*), the characteristic function of $Y$ is

$$\Psi_Y(\omega) = E[e^{j\omega(a_1 X_1 + \cdots + a_n X_n)}] = \Psi_{X_1 \cdots X_n}(\omega a_1, \ldots, \omega a_n) \tag{5.173}$$

Now, by the results of Prob. 4.55, we see that $Y$ is a normal r.v. with mean and variance given by [Eqs. (*4.108*) and (*4.111*)]

$$\mu_Y = E(Y) = \sum_{i=1}^{n} a_i E(X_i) = \sum_{i=1}^{n} a_i \mu_i \tag{5.174}$$

$$\sigma_Y^2 = \mathrm{Var}(Y) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k \, \mathrm{Cov}(X_i, X_k) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k \sigma_{ik} \tag{5.175}$$

Thus, by Eq. (*4.125*),

$$\begin{aligned} \Psi_Y(\omega) &= \exp[j\omega\mu_Y - \tfrac{1}{2}\sigma_Y^2 \omega^2] \\ &= \exp\left( j\omega \sum_{i=1}^{n} a_i \mu_i - \tfrac{1}{2}\omega^2 \sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k \sigma_{ik} \right) \end{aligned} \tag{5.176}$$

Equating Eqs. (*5.176*) and (*5.173*) and setting $\omega = 1$, we get

$$\Psi_{X_1 \cdots X_n}(a_1, \ldots, a_n) = \exp\left( j \sum_{i=1}^{n} a_i \mu_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} a_i a_k \sigma_{ik} \right)$$

By replacing $a_i$'s with $\omega_i$'s, we obtain Eq. (*5.172*); that is,

$$\Psi_{X_1 \cdots X_n}(\omega_1, \ldots, \omega_n) = \exp\left( j \sum_{i=1}^{n} \omega_i \mu_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_i \omega_k \sigma_{ik} \right)$$

Let $$\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} \qquad K = [\sigma_{ik}] = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

Then we can write

$$\sum_{i=1}^{n} \omega_i \mu_i = \boldsymbol{\omega}^T \boldsymbol{\mu} \qquad \sum_{i=1}^{n} \sum_{k=1}^{n} \omega_i \omega_k \sigma_{ik} = \boldsymbol{\omega}^T K \boldsymbol{\omega}$$

and Eq. (*5.172*) can be expressed more compactly as

$$\Psi_{X_1 \cdots X_n}(\omega_1, \ldots, \omega_n) = \exp(j\boldsymbol{\omega}^T \boldsymbol{\mu} - \tfrac{1}{2}\boldsymbol{\omega}^T K \boldsymbol{\omega}) \tag{5.177}$$

**5.60.** Let $X_1, \ldots, X_n$ be jointly normal r.v.'s Let

$$\begin{aligned} Y_1 &= a_{11} X_1 + \cdots + a_{1n} X_n \\ &\vdots \\ Y_m &= a_{m1} X_1 + \cdots + a_{mn} X_n \end{aligned} \tag{5.178}$$

where $a_{ik}$ $(i = 1, \ldots, m; j = 1, \ldots, n)$ are constants. Show that $Y_1, \ldots, Y_m$ are also jointly normal r.v.'s.

Let $$\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \qquad \mathbf{Y} = \begin{bmatrix} Y_1 \\ \vdots \\ Y_m \end{bmatrix} \qquad A = [a_{ik}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

Then Eq. (*5.178*) can be expressed as

$$\mathbf{Y} = A\mathbf{X} \tag{5.179}$$

Let $$\boldsymbol{\mu}_{\mathbf{X}} = E(\mathbf{X}) = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} \qquad \boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_m \end{bmatrix} \qquad K_{\mathbf{X}} = [\sigma_{ik}] = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix}$$

Then the characteristic function for $\mathbf{Y}$ can be written as

$$\begin{aligned} \Psi_{\mathbf{Y}}(\omega_1, \ldots, \omega_m) &= E(e^{j\boldsymbol{\omega}^T\mathbf{Y}}) = E(e^{j\boldsymbol{\omega}^T A\mathbf{X}}) \\ &= E[e^{j(A^T\boldsymbol{\omega})^T\mathbf{X}}] = \Psi_{\mathbf{X}}(A^T\boldsymbol{\omega}) \end{aligned}$$

Since $\mathbf{X}$ is a normal random vector, by Eq. (*5.177*) we can write

$$\begin{aligned} \Psi_{\mathbf{X}}(A^T\boldsymbol{\omega}) &= \exp[j(A^T\boldsymbol{\omega})^T\boldsymbol{\mu}_{\mathbf{X}} - \tfrac{1}{2}(A^T\boldsymbol{\omega})^T K_{\mathbf{X}}(A^T\boldsymbol{\omega})] \\ &= \exp[j\boldsymbol{\omega}^T A\boldsymbol{\mu}_{\mathbf{X}} - \tfrac{1}{2}\boldsymbol{\omega}^T A K_{\mathbf{X}} A^T\boldsymbol{\omega}] \end{aligned}$$

Thus $$\Psi_{\mathbf{Y}}(\omega_1, \ldots, \omega_m) = \exp(j\boldsymbol{\omega}^T\boldsymbol{\mu}_{\mathbf{Y}} - \tfrac{1}{2}\boldsymbol{\omega}^T K_{\mathbf{Y}}\boldsymbol{\omega}) \tag{5.180}$$

where $$\boldsymbol{\mu}_{\mathbf{Y}} = A\boldsymbol{\mu}_{\mathbf{X}} \qquad K_{\mathbf{Y}} = A K_{\mathbf{X}} A^T \tag{5.181}$$

Comparing Eqs. (*5.177*) and (*5.180*), we see that Eq. (*5.180*) is the characteristic function of a random vector $\mathbf{Y}$. Hence, we conclude that $Y_1, \ldots, Y_m$ are also jointly normal r.v.'s

Note that on the basis of the above result, we can say that a random process $\{X(t), t \in T\}$ is a normal process if every finite linear combination of the r.v.'s $X(t_i)$, $t_i \in T$ is normally distributed.

**5.61.** Show that a Wiener process $X(t)$ is a normal process.

Consider an arbitrary linear combination

$$\sum_{i=1}^{n} a_i X(t_i) = a_1 X(t_1) + a_2 X(t_2) + \cdots + a_n X(t_n) \tag{5.182}$$

where $0 \leq t_1 < \cdots < t_n$ and $a_i$ are real constants. Now we write

$$\begin{aligned} \sum_{i=1}^{n} a_i X(t_i) &= (a_1 + \cdots + a_n)[X(t_1) - X(0)] + (a_2 + \cdots + a_n)[X(t_2) - X(t_1)] \\ &\quad + \cdots + (a_{n-1} + a_n)[X(t_{n-1}) - X(t_{n-2})] + a_n[X(t_n) - X(t_{n-1})] \end{aligned} \tag{5.183}$$

Now from conditions 1 and 2 of Definition 5.7.1, the right-hand side of Eq. (*5.183*) is a linear combination of independent normal r.v.'s. Thus, based on the result of Prob. 5.60, the left-hand side of Eq. (*5.183*) is also a normal r.v.; that is, every finite linear combination of the r.v.'s $X(t_i)$ is a normal r.v. Thus we conclude that the Wiener process $X(t)$ is a normal process.

**5.62.** A random process $\{X(t), t \in T\}$ is said to be *continuous in probability* if for every $\varepsilon > 0$ and $t \in T$,

$$\lim_{h \to 0} P\{|X(t + h) - X(t)| > \varepsilon\} = 0 \tag{5.184}$$

Show that a Wiener process $X(t)$ is continuous in probability.

From Chebyshev inequality (*2.97*), we have

$$P\{|X(t + h) - X(t)| > \varepsilon\} \leq \frac{\mathrm{Var}[X(t + h) - X(t)]}{\varepsilon^2} \qquad \varepsilon > 0$$

Since $X(t)$ has stationary increments, we have

$$\text{Var}[X(t+h) - X(t)] = \text{Var}[X(h)] = \sigma^2 h$$

in view of Eq. (5.63). Hence,

$$\lim_{h \to 0} P\{|X(t+h) - X(t)| > \varepsilon\} = \lim_{h \to 0} \frac{\sigma^2 h}{\varepsilon^2} = 0$$

Thus the Wiener process $X(t)$ is continuous in probability.

## Supplementary Problems

**5.63.** Consider a random process $X(n) = \{X_n, n \geq 1\}$, where

$$X_n = Z_1 + Z_2 + \cdots + Z_n$$

and $Z_n$ are iid r.v.'s with zero mean and variance $\sigma^2$. Is $X(n)$ stationary?

*Ans.* No.

**5.64.** Consider a random process $X(t)$ defined by

$$X(t) = Y \cos(\omega t + \Theta)$$

where $Y$ and $\Theta$ are independent r.v.'s and are uniformly distributed over $(-A, A)$ and $(-\pi, \pi)$, respectively.

(*a*) Find the mean of $X(t)$.

(*b*) Find the autocorrelation function $R_X(t, s)$ of $X(t)$.

*Ans.* (*a*) $E[X(t)] = 0$; (*b*) $R_X(t, s) = \frac{1}{6}A^2 \cos \omega(t - s)$

**5.65.** Suppose that a random process $X(t)$ is wide-sense stationary with autocorrelation

$$R_X(t, t+\tau) = e^{-|\tau|/2}$$

(*a*) Find the second moment of the r.v. $X(5)$.

(*b*) Find the second moment of the r.v. $X(5) - X(3)$.

*Ans.* (*a*) $E[X^2(5)] = 1$; (*b*) $E\{[X(5) - X(3)]^2\} = 2(1 - e^{-1})$

**5.66.** Consider a random process $X(t)$ defined by

$$X(t) = U \cos t + (V + 1) \sin t \qquad -\infty < t < \infty$$

where $U$ and $V$ are independent r.v.'s for which

$$E(U) = E(V) = 0 \qquad E(U^2) = E(V^2) = 1$$

(*a*) Find the autocovariance function $K_X(t, s)$ of $X(t)$.

(*b*) Is $X(t)$ WSS?

*Ans.* (*a*) $K_X(t, s) = \cos(s - t)$; (*b*) No.

**5.67.** Consider the random processes

$$X(t) = A_0 \cos(\omega_0 t + \Theta) \qquad Y(t) = A_1 \cos(\omega_1 t + \Phi)$$

where $A_0$, $A_1$, $\omega_0$, and $\omega_1$ are constants, and r.v.'s $\Theta$ and $\Phi$ are independent and uniformly distributed over $(-\pi, \pi)$.

(*a*) Find the cross-correlation function of $R_{XY}(t, t+\tau)$ of $X(t)$ and $Y(t)$.

(*b*) Repeat (*a*) if $\Theta = \phi$.

*Ans.* (*a*) $R_{XY}(t, t+\tau)] = 0$

(*b*) $R_{XY}(t, t+\tau) = \dfrac{A_0 A_1}{2} \cos[(\omega_1 - \omega_0)t + \omega_1 \tau]$

**5.68.** Given a Markov chain $\{X_n, n \geq 0\}$, find the joint pmf

$$P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n)$$

*Hint:* Use Eq. (*5.32*).

*Ans.* $p_{i_0}(0)p_{i_0 i_1}p_{i_1 i_2} \cdots p_{i_{n-1} i_n}$

**5.69.** Let $\{X_n, n \geq 0\}$ be a homogeneous Markov chain. Show that

$$P(X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_0 = i_0, \ldots, X_n = i) = P(X_1 = k_1, \ldots, X_m = k_m | X_0 = i)$$

*Hint:* Use the Markov property (*5.27*) and the homogeneity property.

**5.70.** Verify Eq. (*5.37*).

*Hint:* Write Eq. (*5.39*) in terms of components.

**5.71.** Find $P^n$ for the following transition probability matrices:

$$(a)\ P = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix} \qquad (b)\ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (c)\ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.3 & 0.2 & 0.5 \end{bmatrix}$$

*Ans.* (*a*) $P^n = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + (0.5)^n \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$ (*b*) $P^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(*c*) $P^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.6 & 0.4 & 0 \end{bmatrix} + (0.5)^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -0.6 & -0.4 & 1 \end{bmatrix}$

**5.72.** A certain product is made by two companies, A and B, that control the entire market. Currently, A and B have 60 percent and 40 percent, respectively, of the total market. Each year, A loses $\frac{2}{3}$ of its market share to B, while B loses $\frac{1}{2}$ of its share to A. Find the relative proportion of the market that each hold after 2 years.

*Ans.* A has 43.3 percent and B has 56.7 percent.

**5.73.** Consider a Markov chain with state $\{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

Is state 0 periodic?

*Hint:* Draw the state transition diagram.

*Ans.* No.

**5.74.** Verify Eq. (*5.51*).

*Hint:* Let $\tilde{N} = [N_{jk}]$, where $N_{jk}$ is the number of times the state $k(\in B)$ is occupied until absorption takes place when $X(n)$ starts in state $j(\in B)$. Then $T_j = \sum_{k=m+1}^{N} N_{jk}$; calculate $E(N_{jk})$.

**5.75.** Consider a Markov chain with transition probability matrix

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0 & 0.4 \end{bmatrix}$$

Find the steady-state probabilities.

*Ans.* $\hat{\mathbf{p}} = [\frac{5}{9} \quad \frac{2}{9} \quad \frac{2}{9}]$

**5.76.** Let $X(t)$ be a Poisson process with rate $\lambda$. Find $E[X^2(t)]$.

*Ans.* $\lambda t + \lambda^2 t^2$

**5.77.** Let $X(t)$ be a Poisson process with rate $\lambda$. Find $E\{[X(t) - X(s)]^2\}$ for $t > s$.

*Hint:* Use the independent stationary increments condition and the result of Prob. 5.76.

*Ans.* $\lambda(t - s) + \lambda^2(t - s)^2$

**5.78.** Let $X(t)$ be a Poisson process with rate $\lambda$. Find

$$P[X(t - d) = k \mid X(t) = j] \qquad d > 0$$

*Ans.* $$\frac{j!}{k!(j-k)!}\left(\frac{t-d}{t}\right)^k\left(\frac{d}{t}\right)^{j-k}$$

**5.79.** Let $T_n$ denote the time of the $n$th event of a Poisson process with rate $\lambda$. Find the variance of $T_n$.

*Ans.* $n/\lambda^2$

**5.80.** Assume that customers arrive at a bank in accordance with a Poisson process with rate $\lambda = 6$ per hour, and suppose that each customer is a man with probability $\frac{2}{3}$ and a woman with probability $\frac{1}{3}$. Now suppose that 10 men arrived in the first 2 hours. How many woman would you expect to have arrived in the first 2 hours?

*Ans.* 4

**5.81.** Let $X_1, \ldots, X_n$ be jointly normal r.v.'s. Let

$$Y_i = X_i + c_i \qquad i = 1, \ldots, n$$

where $c_i$ are constants. Show that $Y_1, \ldots, Y_n$ are also jointly normal r.v.'s.

*Hint:* See Prob. 5.60.

**5.82.** Derive Eq. (*5.63*).

*Hint:* Use condition (1) of a Wiener process and Eq. (*5.102*) of Prob. 5.22.

# Chapter 6

# Analysis and Processing of Random Processes

## 6.1 INTRODUCTION

In this chapter, we introduce the methods for analysis and processing of random processes. First, we introduce the definitions of stochastic continuity, stochastic derivatives, and stochastic integrals of random processes. Next, the notion of power spectral density is introduced. This concept enables us to study wide-sense stationary processes in the frequency domain and define a white noise process. The response of linear systems to random processes is then studied. Finally, orthogonal and spectral representations of random processes are presented.

## 6.2 CONTINUITY, DIFFERENTIATION, INTEGRATION

In this section, we shall consider only the continuous-time random processes.

### A. Stochastic Continuity:

A random process $X(t)$ is said to be *continuous in mean square* or *mean square (m.s.) continuous* if

$$\lim_{\varepsilon \to 0} E\{[X(t+\varepsilon) - X(t)]^2\} = 0 \tag{6.1}$$

The random process $X(t)$ is m.s. continuous if and only if its autocorrelation function is continuous (Prob. 6.1). If $X(t)$ is WSS, then it is m.s. continuous if and only if its autocorrelation function $R_X(\tau)$ is continuous at $\tau = 0$. If $X(t)$ is m.s. continuous, then its mean is continuous; that is,

$$\lim_{\varepsilon \to 0} \mu_X(t+\varepsilon) = \mu_X(t) \tag{6.2}$$

which can be written as

$$\lim_{\varepsilon \to 0} E[X(t+\varepsilon)] = E[\lim_{\varepsilon \to 0} X(t+\varepsilon)] \tag{6.3}$$

Hence, if $X(t)$ is m.s. continuous, then we may interchange the ordering of the operations of expectation and limiting. Note that m.s. continuity of $X(t)$ does not imply that the sample functions of $X(t)$ are continuous. For instance, the Poisson process is m.s. continuous (Prob. 6.46), but sample functions of the Poisson process have a countably infinite number of discontinuities (see Fig. 5-2).

### B. Stochastic Derivatives:

A random process $X(t)$ is said to have a *m.s. derivative* $X'(t)$ if

$$\text{l.i.m.}_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon} = X'(t) \tag{6.4}$$

where l.i.m. denotes *l*imit *i*n the *m*ean (square); that is,

$$\lim_{\varepsilon \to 0} E\left\{\left[\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right]^2\right\} = 0 \tag{6.5}$$

The m.s. derivative of $X(t)$ exists if $\partial^2 R_X(t, s)/\partial t\ \partial s$ exists (Prob. 6.6). If $X(t)$ has the m.s. derivative $X'(t)$, then its mean and autocorrelation function are given by

$$E[X'(t)] = \frac{d}{dt} E[X(t)] = \mu'_X(t) \tag{6.6}$$

$$R_{X'}(t, s) = \frac{\partial^2 R_X(t, s)}{\partial t\ \partial s} \tag{6.7}$$

Equation (6.6) indicates that the operations of differentiation and expectation may be interchanged. If $X(t)$ is a normal random process for which the m.s. derivative $X'(t)$ exists, then $X'(t)$ is also a normal random process (Prob. 6.10).

### C. Stochastic Integrals:

A *m.s. integral* of a random process $X(t)$ is defined by

$$Y(t) = \int_{t_0}^{t} X(\alpha)\, d\alpha = \text{l.i.m.}_{\Delta t_i \to 0} \sum_i X(t_i)\, \Delta t_i \tag{6.8}$$

where $t_0 < t_1 < \cdots < t$ and $\Delta t_i = t_{i+1} - t_i$.

The m.s. integral of $X(t)$ exists if the following integral exists (Prob. 6.11):

$$\int_{t_0}^{t} \int_{t_0}^{t} R_X(\alpha, \beta)\, d\alpha\, d\beta \tag{6.9}$$

This implies that if $X(t)$ is m.s. continuous, then its m.s. integral $Y(t)$ exists (see Prob. 6.1). The mean and the autocorrelation function of $Y(t)$ are given by

$$\mu_Y(t) = E\left[\int_{t_0}^{t} X(\alpha)\, d\alpha\right] = \int_{t_0}^{t} E[X(\alpha)]\, d\alpha = \int_{t_0}^{t} \mu_X(\alpha)\, d\alpha \tag{6.10}$$

$$\begin{aligned} R_Y(t, s) &= E\left[\int_{t_0}^{t} X(\alpha)\, d\alpha \int_{t_0}^{s} X(\beta)\, d\beta\right] \\ &= \int_{t_0}^{t} \int_{t_0}^{s} E[X(\alpha)X(\beta)]\, d\beta\, d\alpha = \int_{t_0}^{t} \int_{t_0}^{s} R_X(\alpha, \beta)\, d\beta\, d\alpha \end{aligned} \tag{6.11}$$

Equation (6.10) indicates that the operations of integration and expectation may be interchanged. If $X(t)$ is a normal random process, then its integral $Y(t)$ is also a normal random process. This follows from the fact that $\Sigma_i X(t_i)\, \Delta t_i$ is a linear combination of the jointly normal r.v.'s. (see Prob. 5.60).

## 6.3 POWER SPECTRAL DENSITIES

In this section we assume that all random processes are WSS.

### A. Autocorrelation Functions:

The autocorrelation function of a continuous-time random process $X(t)$ is defined as [Eq. (5.7)]

$$R_X(\tau) = E[X(t)X(t + \tau)] \tag{6.12}$$

**Properties of $R_X(\tau)$:**

1. $R_X(-\tau) = R_X(\tau)$ (6.13)
2. $|R_X(\tau)| \leq R_X(0)$ (6.14)
3. $R_X(0) = E[X^2(t)] \geq 0$ (6.15)

Property 3 [Eq. (6.15)] is easily obtained by setting $\tau = 0$ in Eq. (6.12). If we assume that $X(t)$ is a voltage waveform across a 1-Ω resistor, then $E[X^2(t)]$ is the average value of power delivered to the 1-Ω resistor by $X(t)$. Thus, $E[X^2(t)]$ is often called the *average power* of $X(t)$. Properties 1 and 2 are verified in Prob. 6.13.

In case of a discrete-time random process $X(n)$, the autocorrelation function of $X(n)$ is defined by

$$R_X(k) = E[X(n)X(n+k)] \tag{6.16}$$

Various properties of $R_X(k)$ similar to those of $R_X(\tau)$ can be obtained by replacing $\tau$ by $k$ in Eqs. (6.13) to (6.15).

### B. Cross-Correlation Functions

The cross-correlation function of two continuous-time jointly WSS random processes $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)] \tag{6.17}$$

**Properties of $R_{XY}(\tau)$:**

1. $R_{XY}(-\tau) = R_{YX}(\tau)$ (6.18)
2. $|R_{XY}(\tau)| \leq \sqrt{R_X(0)R_Y(0)}$ (6.19)
3. $|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$ (6.20)

These properties are verified in Prob. 6.14. Two processes $X(t)$ and $Y(t)$ are called (*mutually*) *orthogonal* if

$$R_{XY}(\tau) = 0 \qquad \text{for all } \tau \tag{6.21}$$

Similarly, the cross-correlation function of two discrete-time jointly WSS random processes $X(n)$ and $Y(n)$ is defined by

$$R_{XY}(k) = E[X(n)Y(n+k)] \tag{6.22}$$

and various properties of $R_{XY}(k)$ similar to those of $R_{XY}(\tau)$ can be obtained by replacing $\tau$ by $k$ in Eqs. (6.18) to (6.20).

### C. Power Spectral Density:

The *power spectral density* (or *power spectrum*) $S_X(\omega)$ of a continuous-time random process $X(t)$ is defined as the Fourier transform of $R_X(\tau)$:

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\, d\tau \tag{6.23}$$

Thus, taking the inverse Fourier transform of $S_X(\omega)$, we obtain

$$R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}\, d\omega \tag{6.24}$$

Equations (6.23) and (6.24) are known as the *Wiener-Khinchin relations.*

***Properties of $S_X(\omega)$:***

1. $S_X(\omega)$ is real and $S_X(\omega) \geq 0$. (6.25)

2. $S_X(-\omega) = S_X(\omega)$ (6.26)

3. $E[X^2(t)] = R_X(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_X(\omega)\, d\omega$ (6.27)

Similarly, the power spectral density $S_X(\Omega)$ of a discrete-time random process $X(n)$ is defined as the Fourier transform of $R_X(k)$:

$$S_X(\Omega) = \sum_{k=-\infty}^{\infty} R_X(k)e^{-j\Omega k} \tag{6.28}$$

Thus, taking the inverse Fourier transform of $S_X(\Omega)$, we obtain

$$R_X(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(\Omega)e^{j\Omega k}\, d\Omega \tag{6.29}$$

***Properties of $S_X(\Omega)$:***

1. $S_X(\Omega + 2\pi) = S_X(\Omega)$ (6.30)
2. $S_X(\Omega)$ is real and $S_X(\Omega) \geq 0$. (6.31)
3. $S_X(-\Omega) = S_X(\Omega)$ (6.32)
4. $E[X^2(n)] = R_X(0) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} S_X(\Omega)\, d\Omega$ (6.33)

Note that property 1 [Eq. (6.30)] follows from the fact that $e^{-j\Omega k}$ is periodic with period $2\pi$. Hence it is sufficient to define $S_X(\Omega)$ only in the range $(-\pi, \pi)$.

**D. Cross Power Spectral Densities:**

The *cross power spectral density* (or *cross power spectrum*) $S_{XY}(\omega)$ of two continuous-time random processes $X(t)$ and $Y(t)$ is defined as the Fourier transform of $R_{XY}(\tau)$:

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\, d\tau \tag{6.34}$$

Thus, taking the inverse Fourier transform of $S_{XY}(\omega)$, we get

$$R_{XY}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega\tau}\, d\omega \tag{6.35}$$

***Properties of $S_{XY}(\omega)$:***

Unlike $S_X(\omega)$, which is a real-valued function of $\omega$, $S_{XY}(\omega)$, in general, is a complex-valued function.

1. $S_{XY}(\omega) = S_{YX}(-\omega)$ (6.36)
2. $S_{XY}(-\omega) = S_{XY}^*(\omega)$ (6.37)

Similarly, the cross power spectral density $S_{XY}(\Omega)$ of two discrete-time random processes $X(n)$ and $Y(n)$ is defined as the Fourier transform of $R_{XY}(k)$:

$$S_{XY}(\Omega) = \sum_{k=-\infty}^{\infty} R_{XY}(k)e^{-j\Omega k} \tag{6.38}$$

Thus, taking the inverse Fourier transform of $S_{XY}(\Omega)$, we get

$$R_{XY}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{XY}(\Omega)e^{j\Omega k}\, d\Omega \tag{6.39}$$

***Properties of $S_{XY}(\Omega)$:***

Unlike $S_X(\Omega)$, which is a real-valued function of $\omega$, $S_{XY}(\Omega)$, in general, is a complex-valued function.

1. $S_{XY}(\Omega + 2\pi) = S_{XY}(\Omega)$ (6.40)
2. $S_{XY}(\Omega) = S_{YX}(-\Omega)$ (6.41)
3. $S_{XY}(-\Omega) = S_{XY}^*(\Omega)$ (6.42)

## 6.4 WHITE NOISE

A continuous-time *white noise* process, $W(t)$, is a WSS zero-mean continuous-time random process whose autocorrelation function is given by

$$R_W(\tau) = \sigma^2\delta(\tau) \tag{6.43}$$

where $\delta(\tau)$ is a unit impulse function (or Dirac $\delta$ function) defined by

$$\int_{-\infty}^{\infty} \delta(\tau)\phi(\tau)\, d\tau = \phi(0) \tag{6.44}$$

where $\phi(\tau)$ is any function continuous at $\tau = 0$. Taking the Fourier transform of Eq. (6.43), we obtain

$$S_W(\omega) = \sigma^2 \int_{-\infty}^{\infty} \delta(\tau)e^{-j\omega\tau}\, d\tau = \sigma^2 \tag{6.45}$$

which indicates that $X(t)$ has a constant power spectral density (hence the name white noise). Note that the average power of $W(t)$ is not finite.

Similarly, a WSS zero-mean discrete-time random process $W(n)$ is called a discrete-time white noise if its autocorrelation function is given by

$$R_W(k) = \sigma^2\delta(k) \tag{6.46}$$

where $\delta(k)$ is a unit impulse sequence (or unit sample sequence) defined by

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{6.47}$$

Taking the Fourier transform of Eq. (6.46), we obtain

$$S_W(\Omega) = \sigma^2 \sum_{k=-\infty}^{\infty} \delta(k)e^{-j\Omega k} = \sigma^2 \qquad -\pi < \Omega < \pi \tag{6.48}$$

Again the power spectral density of $W(n)$ is a constant. Note that $S_W(\Omega + 2\pi) = S_W(\Omega)$ and the average power of $W(n)$ is $\sigma^2 = \mathrm{Var}[W(n)]$, which is finite.

## 6.5 RESPONSE OF LINEAR SYSTEMS TO RANDOM INPUTS

### A. Linear Systems:

A system is a mathematical model of a physical process that relates the input (or excitation) signal $x$ to the output (or response) signal $y$. Then the system is viewed as a transformation (or mapping) of $x$ into $y$. This transformation is represented by the operator $\mathbf{T}$ as (Fig. 6-1)

$$y = \mathbf{T}x \tag{6.49}$$

If $x$ and $y$ are continuous-time signals, then the system is called a continuous-time system, and if $x$ and $y$ are discrete-time signals, then the system is called a discrete-time system. If the operator $\mathbf{T}$ is a

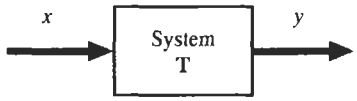



Fig. 6-1

linear operator satisfying

$$\mathbf{T}\{x_1 + x_2\} = \mathbf{T}x_1 + \mathbf{T}x_2 = y_1 + y_2 \qquad \text{(Additivity)}$$
$$\mathbf{T}\{\alpha x\} = \alpha \mathbf{T}x = \alpha y \qquad \text{(Homogeneity)}$$

where $\alpha$ is a scalar number, then the system represented by $\mathbf{T}$ is called a linear system. A system is called time-invariant if a time shift in the input signal causes the same time shift in the output signal. Thus, for a continuous-time system,

$$\mathbf{T}\{x(t - t_0)\} = y(t - t_0)$$

for any value of $t_0$, and for a discrete-time system,

$$\mathbf{T}\{x(n - n_0)\} = y(n - n_0)$$

for any integer $n_0$. For a continuous-time linear time-invariant (LTI) system, Eq. (*6.49*) can be expressed as

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t - \lambda)\, d\lambda \tag{6.50}$$

where

$$h(t) = \mathbf{T}\{\delta(t)\} \tag{6.51}$$

is known as the *impulse response* of a continuous-time LTI system. The right-hand side of Eq. (*6.50*) is commonly called the *convolution integral* of $h(t)$ and $x(t)$, denoted by $h(t) * x(t)$. For a discrete-time LTI system, Eq. (*6.49*) can be expressed as

$$y(n) = \sum_{i=-\infty}^{\infty} h(i)x(n - i) \tag{6.52}$$

where

$$h(n) = \mathbf{T}\{\delta(n)\} \tag{6.53}$$

is known as the *impulse response* (or *unit sample response*) of a discrete-time LTI system. The right-hand side of Eq. (*6.52*) is commonly called the *convolution sum* of $h(n)$ and $x(n)$, denoted by $h(n) * x(n)$.

### B. Response of a Continuous-Time Linear System to Random Input:

When the input to a continuous-time linear system represented by Eq. (*6.49*) is a random process $\{X(t), t \in T_x\}$, then the output will also be a random process $\{Y(t), t \in T_y\}$; that is,

$$\mathbf{T}\{X(t), t \in T_x\} = \{Y(t), t \in T_y\} \tag{6.54}$$

For any input sample function $x_i(t)$, the corresponding output sample function is

$$y_i(t) = \mathbf{T}\{x_i(t)\} \tag{6.55}$$

If the system is LTI, then by Eq. (*6.50*), we can write

$$Y(t) = \int_{-\infty}^{\infty} h(\lambda)X(t - \lambda)\, d\lambda \tag{6.56}$$

Note that Eq. (*6.56*) is a stochastic integral. Then

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\lambda)E[X(t - \lambda)]\, d\lambda \tag{6.57}$$

The autocorrelation function of $Y(t)$ is given by (Prob. 6.24)

$$R_Y(t, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(t - \alpha, s - \beta)\, d\alpha\, d\beta \tag{6.58}$$

If the input $X(t)$ is WSS, then from Eq. (6.57),

$$E[Y(t)] = \mu_X \int_{-\infty}^{\infty} h(\lambda)\, d\lambda = \mu_X H(0) \tag{6.59}$$

where $H(0) = H(\omega)|_{\omega=0}$ and $H(\omega)$ is the frequency response of the system defined by the Fourier transform of $h(t)$; that is,

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}\, dt \tag{6.60}$$

The autocorrelation function of $Y(t)$ is, from Eq. (6.58),

$$R_Y(t, s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(s - t + \alpha - \beta)\, d\alpha\, d\beta \tag{6.61}$$

Setting $s = t + \tau$, we get

$$R_Y(t, t + \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau + \alpha - \beta)\, d\alpha\, d\beta = R_Y(\tau) \tag{6.62}$$

From Eqs. (6.59) and (6.62), we see that the output $Y(t)$ is also WSS. Taking the Fourier transform of Eq. (6.62), the power spectral density of $Y(t)$ is given by (Prob. 6.25)

$$S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau)e^{-j\omega\tau}\, d\tau = |H(\omega)|^2 S_X(\omega) \tag{6.63}$$

Thus, we obtain the important result that *the power spectral density of the output is the product of the power spectral density of the input and the magnitude squared of the frequency response of the system.*

When the autocorrelation function of the output $R_Y(\tau)$ is desired, it is easier to determine the power spectral density $S_Y(\omega)$ and then evaluate the inverse Fourier transform (Prob. 6.26). Thus,

$$R_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega)e^{j\omega\tau}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega)e^{j\omega\tau}\, d\omega \tag{6.64}$$

By Eq. (6.15), the average power in the output $Y(t)$ is

$$E[Y^2(t)] = R_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega)\, d\omega \tag{6.65}$$

### C. Response of a Discrete-Time Linear System to Random Input:

When the input to a discrete-time LTI system is a discrete-time random process $X(n)$, then by Eq. (6.52), the output $Y(n)$ is

$$Y(n) = \sum_{i=-\infty}^{\infty} h(i)X(n - i) \tag{6.66}$$

The autocorrelation function of $Y(n)$ is given by

$$R_Y(n, m) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(n - i, m - l) \tag{6.67}$$

When $X(n)$ is WSS, then from Eq. (6.66),

$$E[Y(n)] = \mu_X \sum_{i=-\infty}^{\infty} h(i) = \mu_X H(0) \tag{6.68}$$

where $H(0) = H(\Omega)|_{\Omega=0}$ and $H(\Omega)$ is the frequency response of the system defined by the Fourier transform of $h(n)$:

$$H(\Omega) = \sum_{n=-\infty}^{\infty} h(n)e^{-j\Omega n} \tag{6.69}$$

The autocorrelation function of $Y(n)$ is, from Eq. (6.67),

$$R_Y(n, m) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(m - n + i - l) \tag{6.70}$$

Setting $m = n + k$, we get

$$R_Y(n, n + k) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(k + i - l) = R_Y(k) \tag{6.71}$$

From Eqs. (6.68) and (6.71), we see that the output $Y(n)$ is also WSS. Taking the Fourier transform of Eq. (6.71), the power spectral density of $Y(n)$ is given by (Prob. 6.28)

$$S_Y(\Omega) = |H(\Omega)|^2 S_X(\Omega) \tag{6.72}$$

which is the same as Eq. (6.63).

## 6.6 FOURIER SERIES AND KARHUNEN-LOÉVE EXPANSIONS

### A. Stochastic Periodicity:

A continuous-time random process $X(t)$ is said to be *m.s. periodic* with period $T$ if

$$E\{[X(t + T) - X(t)]^2\} = 0 \tag{6.73}$$

If $X(t)$ is WSS, then $X(t)$ is m.s. periodic if and only if its autocorrelation function is periodic with period $T$; that is,

$$R_X(\tau + T) = R_X(\tau) \tag{6.74}$$

### B. Fourier Series:

Let $X(t)$ be a WSS random process with periodic $R_X(\tau)$ having period $T$. Expanding $R_X(\tau)$ into a Fourier series, we obtain

$$R_X(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0\tau} \qquad \omega_0 = 2\pi/T \tag{6.75}$$

where

$$c_n = \frac{1}{T}\int_0^T R_X(\tau)e^{-jn\omega_0\tau}\, d\tau \tag{6.76}$$

Let $\hat{X}(t)$ be expressed as

$$\hat{X}(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad \omega_0 = 2\pi/T \tag{6.77}$$

where $X_n$ are r.v.'s given by

$$X_n = \frac{1}{T}\int_0^T X(t)e^{-jn\omega_0 t}\,dt \tag{6.78}$$

Note that, in general, $X_n$ are complex-valued r.v.'s. For complex-valued r.v.'s, the correlation between two r.v.'s $X$ and $Y$ is defined by $E(XY^*)$. Then $\hat{X}(t)$ is called the m.s. Fourier series of $X(t)$ such that (Prob. 6.34)

$$E\{|X(t) - \hat{X}(t)|^2\} = 0 \tag{6.79}$$

Furthermore, we have (Prob. 6.33)

$$E(X_n) = \mu_X \delta(n) = \begin{cases} \mu_X & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{6.80}$$

$$E(X_n X_m^*) = c_n \delta(n - m) = \begin{cases} c_n & n = m \\ 0 & n \neq m \end{cases} \tag{6.81}$$

### C. Karhunen-Loéve Expansion

Consider a random process $X(t)$ which is not periodic. Let $\hat{X}(t)$ be expressed as

$$\hat{X}(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \qquad 0 < t < T \tag{6.82}$$

where a set of functions $\{\phi_n(t)\}$ is orthonormal on an interval $(0, T)$ such that

$$\int_0^T \phi_n(t)\phi_m^*(t)\,dt = \delta(n - m) \tag{6.83}$$

and $X_n$ are r.v.'s given by

$$X_n = \int_0^T X(t)\phi_n^*(t)\,dt \tag{6.84}$$

Then $\hat{X}(t)$ is called the *Karhunen-Loéve expansion* of $X(t)$ such that (Prob. 6.38)

$$E\{|X(t) - \hat{X}(t)|^2\} = 0 \tag{6.85}$$

Let $R_X(t, s)$ be the autocorrelation function of $X(t)$, and consider the following integral equation:

$$\int_0^T R_X(t, s)\phi_n(s)\,ds = \lambda_n \phi_n(t) \qquad 0 < t < T \tag{6.86}$$

where $\lambda_n$ and $\phi_n(t)$ are called the eigenvalues and the corresponding eigenfunctions of the integral equation (6.86). It is known from the theory of integral equations that if $R_X(t, s)$ is continuous, then $\phi_n(t)$ of Eq. (6.86) are orthonormal as in Eq. (6.83), and they satisfy the following identity:

$$R_X(t, s) = \sum_{n=1}^{\infty} \lambda_n \phi_n(t)\phi_n^*(s) \tag{6.87}$$

which is known as *Mercer's theorem.*

With the above results, we can show that Eq. (6.85) is satisfied and the coefficient $X_n$ are orthogonal r.v.'s (Prob. 6.37); that is,

$$E(X_n X_m^*) = \lambda_n \delta(n - m) = \begin{cases} \lambda_n & n = m \\ 0 & n \neq m \end{cases} \tag{6.88}$$

## 6.7 FOURIER TRANSFORM OF RANDOM PROCESSES

### A. Continuous-Time Random Processes:

The Fourier transform of a continuous-time random process $X(t)$ is a random process $\tilde{X}(\omega)$ given by

$$\tilde{X}(\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt \tag{6.89}$$

which is the stochastic integral, and the integral is interpreted as an m.s. limit; that is,

$$E\left\{\left|\tilde{X}(\omega) - \int_{-\infty}^{\infty} X(t)e^{-j\omega t}\, dt\right|^2\right\} = 0 \tag{6.90}$$

Note that $\tilde{X}(\omega)$ is a complex random process. Similarly, the inverse Fourier transform

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}(\omega)e^{j\omega t}\, d\omega \tag{6.91}$$

is also a stochastic integral and should also be interpreted in the m.s. sense. The properties of continuous-time Fourier transforms (Appendix B) also hold for random processes (or random signals). For instance, if $Y(t)$ is the output of a continuous-time LTI system with input $X(t)$, then

$$\tilde{Y}(\omega) = \tilde{X}(\omega)H(\omega) \tag{6.92}$$

where $H(\omega)$ is the frequency response of the system.

Let $\tilde{R}_X(\omega_1, \omega_2)$ be the two-dimensional Fourier transform of $R_X(t, s)$; that is,

$$\tilde{R}_X(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t, s)e^{-j(\omega_1 t + \omega_2 s)}\, dt\, ds \tag{6.93}$$

Then the autocorrelation function of $\tilde{X}(\omega)$ is given by (Prob. 6.41)

$$R_{\tilde{X}}(\omega_1, \omega_2) = E[\tilde{X}(\omega_1)\tilde{X}^*(\omega_2)] = \tilde{R}_X(\omega_1, -\omega_2) \tag{6.94}$$

If $X(t)$ is real, then

$$E[\tilde{X}(\omega_1)\tilde{X}(\omega_2)] = \tilde{R}_X(\omega_1, \omega_2) \tag{6.95}$$

$$\tilde{X}(-\omega) = \tilde{X}^*(\omega) \tag{6.96}$$

$$\tilde{R}_X(-\omega_1, -\omega_2) = \tilde{R}_X^*(\omega_1, \omega_2) \tag{6.97}$$

If $X(t)$ is a WSS random process with autocorrelation function $R_X(t, s) = R_X(t - s) = R_X(\tau)$ and power spectral density $S_X(\omega)$, then (Prob. 6.42)

$$\tilde{R}_X(\omega_1, \omega_2) = 2\pi S_X(\omega_1)\delta(\omega_1 + \omega_2) \tag{6.98}$$

$$R_{\tilde{X}}(\omega_1, \omega_2) = 2\pi S_X(\omega_1)\delta(\omega_1 - \omega_2) \tag{6.99}$$

Equation (6.99) shows that the Fourier transform of a WSS random process is nonstationary white noise.

### B. Discrete-Time Random Processes:

The Fourier transform of a discrete-time random process $X(n)$ is a random process $\tilde{X}(\Omega)$ given by (in m.s. sense)

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{\infty} X(n)e^{-j\Omega n} \tag{6.100}$$

Similarly, the inverse Fourier transform

$$X(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \tilde{X}(\Omega)e^{j\Omega n}\,d\Omega \tag{6.101}$$

should also be interpreted in the m.s. sense. Note that $\tilde{X}(\Omega + 2\pi) = \tilde{X}(\Omega)$ and the properties of discrete-time Fourier transforms (Appendix B) also hold for discrete-time random signals. For instance, if $Y(n)$ is the output of a discrete-time LTI system with input $X(n)$, then

$$\tilde{Y}(\Omega) = \tilde{X}(\Omega)H(\Omega) \tag{6.102}$$

where $H(\Omega)$ is the frequency response of the system.

Let $\tilde{R}_X(\Omega_1, \Omega_2)$ be the two-dimensional Fourier transform of $R_X(n, m)$:

$$\tilde{R}_X(\Omega_1, \Omega_2) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_X(n, m)e^{-j(\Omega_1 n + \Omega_2 m)} \tag{6.103}$$

Then the autocorrelation function of $\tilde{X}(\Omega)$ is given by (Prob. 6.44)

$$R_{\tilde{X}}(\Omega_1, \Omega_2) = E[\tilde{X}(\Omega_1)\tilde{X}^*(\Omega_2)] = \tilde{R}_X(\Omega_1, -\Omega_2) \tag{6.104}$$

If $X(n)$ is a WSS random process with autocorrelation function $R_X(n, m) = R_X(n - m) = R_X(k)$ and power spectral density $S_X(\Omega)$, then

$$\tilde{R}_X(\Omega_1, \Omega_2) = 2\pi S_X(\Omega_1)\delta(\Omega_1 + \Omega_2) \tag{6.105}$$

$$R_{\tilde{X}}(\Omega_1, \Omega_2) = 2\pi S_X(\Omega_1)\delta(\Omega_1 - \Omega_2) \tag{6.106}$$

Equation (6.106) shows that the Fourier transform of a discrete-time WSS random process is nonstationary white noise.

## Solved Problems

### CONTINUITY, DIFFERENTIATION, INTEGRATION

**6.1.** Show that the random process $X(t)$ is m.s. continuous if and only if its autocorrelation function $R_X(t, s)$ is continuous.

We can write

$$\begin{aligned} E\{[X(t+\varepsilon) - X(t)]^2\} &= E[X^2(t+\varepsilon) - 2X(t+\varepsilon)X(t) + X^2(t)] \\ &= R_X(t+\varepsilon, t+\varepsilon) - 2R_X(t+\varepsilon, t) + R_X(t, t) \end{aligned} \tag{6.107}$$

Thus, if $R_X(t, s)$ is continuous, then

$$\lim_{\varepsilon\to 0} E\{[X(t+\varepsilon) - X(t)]^2\} = \lim_{\varepsilon\to 0}\{R_X(t+\varepsilon, t+\varepsilon) - 2R_X(t+\varepsilon, t) + R_X(t, t)\} = 0$$

and $X(t)$ is m.s. continuous. Next, consider

$$\begin{aligned} R_X(t+\varepsilon_1, t+\varepsilon_2) - R_X(t, t) &= E\{[X(t+\varepsilon_1) - X(t)][X(t+\varepsilon_2) - X(t)]\} \\ &\quad + E\{[X(t+\varepsilon_1) - X(t)]X(t)\} + E\{[X(t+\varepsilon_2) - X(t)]X(t)\} \end{aligned}$$

Applying Cauchy-Schwarz inequality (3.97) (Prob. 3.35), we obtain

$$\begin{aligned} R_X(t+\varepsilon_1, t+\varepsilon_2) - R_X(t, t) &\le (E\{[X(t+\varepsilon_1) - X(t)]^2\}E\{[X(t+\varepsilon_2) - X(t)]^2\})^{1/2} \\ &\quad + (E\{[X(t+\varepsilon_1) - X(t)]^2\}E[X^2(t)])^{1/2} + (E\{[X(t+\varepsilon_2) - X(t)]^2\}E[X^2(t)])^{1/2} \end{aligned}$$

Thus if $X(t)$ is m.s. continuous, then by Eq. (6.1) we have

$$\lim_{\varepsilon_1, \varepsilon_2\to 0} R_X(t+\varepsilon_1, t+\varepsilon_2) - R_X(t, t) = 0$$

that is, $R_X(t, s)$ is continuous. This completes the proof.

**6.2.** Show that a WSS random process $X(t)$ is m.s. continuous if and only if its autocorrelation function $R_X(\tau)$ is continous at $\tau = 0$.

If $X(t)$ is WSS, then Eq. (*6.107*) becomes

$$E\{[X(t+\varepsilon) - X(t)]^2\} = 2[R_X(0) - R_X(\varepsilon)] \tag{6.108}$$

Thus if $R_X(\tau)$ is continuous at $\tau = 0$, that is,

$$\lim_{\varepsilon \to 0} [R_X(\varepsilon) - R_X(0)] = 0$$

then

$$\lim_{\varepsilon \to 0} E\{[X(t+\varepsilon) - X(t)]^2\} = 0$$

that is, $X(t)$ is m.s. continuous. Similarly, we can show that if $X(t)$ is m.s. continuous, then by Eq. (*6.108*), $R_X(\tau)$ is continuous at $\tau = 0$.

**6.3.** Show that if $X(t)$ is m.s. continuous, then its mean is continuous; that is,

$$\lim_{\varepsilon \to 0} \mu_X(t+\varepsilon) = \mu_X(t)$$

We have

$$\mathrm{Var}[X(t+\varepsilon) - X(t)] = E\{[X(t+\varepsilon) - X(t)]^2\} - \{E[X(t+\varepsilon) - X(t)]\}^2 \geq 0$$

Thus

$$E\{[X(t+\varepsilon) - X(t)]^2\} \geq \{E[X(t+\varepsilon) - X(t)]\}^2 = [\mu_X(t+\varepsilon) - \mu_X(t)]^2$$

If $X(t)$ is m.s. continuous, then as $\varepsilon \to 0$, the left-hand side of the above expression approaches zero. Thus

$$\lim_{\varepsilon \to 0} [\mu_X(t+\varepsilon) - \mu_X(t)] = 0 \qquad \text{or} \qquad \lim_{\varepsilon \to 0} [\mu_X(t+\varepsilon) = \mu_X(t)$$

**6.4.** Show that the Wiener process $X(t)$ is m.s. continuous.

From Eq. (*5.64*), the autocorrelation function of the Wiener process $X(t)$ is given by

$$R_X(t, s) = \sigma^2 \min(t, s)$$

Thus, we have

$$|R_X(t+\varepsilon_1, t+\varepsilon_2) - R_X(t, t)| = \sigma^2 |\min(t+\varepsilon_1, t+\varepsilon_2) - t| \leq \sigma^2 \max(\varepsilon_1, \varepsilon_2)$$

Since

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \max(\varepsilon_1, \varepsilon_2) = 0$$

$R_X(t, s)$ is continuous. Hence the Wiener process $X(t)$ is m.s. continuous.

**6.5.** Show that every m.s. continuous random process is continuous in probability.

A random process $X(t)$ is continuous in probability if, for every $t$ and $a > 0$ (see Prob. 5.62),

$$\lim_{\varepsilon \to 0} P\{|X(t+\varepsilon) - X(t)| > a\} = 0$$

Applying Chebyshev inequality (*2.97*) (Prob. 2.37), we have

$$P\{|X(t+\varepsilon) - X(t)| > a\} \leq \frac{E[|X(t+\varepsilon) - X(t)|^2]}{a^2}$$

Now, if $X(t)$ is m.s. continuous, then the right-hand side goes to 0 as $\varepsilon \to 0$, which implies that the left-hand side must also go to 0 as $\varepsilon \to 0$. Thus, we have proved that if $X(t)$ is m.s. continuous, then it is also continuous in probability.

**6.6.** Show that a random process $X(t)$ has a m.s. derivative $X'(t)$ if $\partial^2 R_X(t, s)/\partial t\, \partial s$ exists at $s = t$.

Let
$$Y(t;\varepsilon) = \frac{X(t+\varepsilon) - X(t)}{\varepsilon} \tag{6.109}$$

By the Cauchy criterion (see the note at the end of this solution), the m.s. derivative $X'(t)$ exists if

$$\lim_{\varepsilon_1,\,\varepsilon_2 \to 0} E\{[Y(t;\varepsilon_2) - Y(t;\varepsilon_1)]^2\} = 0 \tag{6.110}$$

Now
$$\begin{aligned} E\{[Y(t;\varepsilon_2) - Y(t;\varepsilon_1)]^2\} &= E[Y^2(t;\varepsilon_2) - 2Y(t;\varepsilon_2)Y(t;\varepsilon_1) + Y^2(t;\varepsilon_1)] \\ &= E[Y^2(t;\varepsilon_2)] - 2E[Y(t;\varepsilon_2)Y(t;\varepsilon_1)] + E[Y^2(t;\varepsilon_1)] \end{aligned} \tag{6.111}$$

and
$$\begin{aligned} E[Y(t;\varepsilon_2)Y(t;\varepsilon_1)] &= \frac{1}{\varepsilon_1\varepsilon_2} E\{[X(t+\varepsilon_2) - X(t)][X(t+\varepsilon_1) - X(t)]\} \\ &= \frac{1}{\varepsilon_1\varepsilon_2}[R_X(t+\varepsilon_2, t+\varepsilon_1) - R_X(t+\varepsilon_2, t) - R_X(t, t+\varepsilon_1) + R_X(t, t)] \\ &= \frac{1}{\varepsilon_2}\left\{\frac{R_X(t+\varepsilon_2, t+\varepsilon_1) - R_X(t+\varepsilon_2, t)}{\varepsilon_1} - \frac{R_X(t, t+\varepsilon_1) - R_X(t, t)}{\varepsilon_1}\right\} \end{aligned}$$

Thus
$$\lim_{\varepsilon_1,\,\varepsilon_2 \to 0} E[Y(t;\varepsilon_2)Y(t;\varepsilon_1)] = \left.\frac{\partial^2 R_X(t, s)}{\partial t\, \partial s}\right|_{s=t} = R_2 \tag{6.112}$$

provided $\partial^2 R_X(t, s)/\partial t\, \partial s$ exists at $s = t$. Setting $\varepsilon_1 = \varepsilon_2$ in Eq. (6.112), we get

$$\lim_{\varepsilon_1 \to 0} E[Y^2(t;\varepsilon_1)] = \lim_{\varepsilon_2 \to 0} E[Y^2(t;\varepsilon_2)] = R_2$$

and by Eq. (6.111), we obtain

$$\lim_{\varepsilon_1,\,\varepsilon_2 \to 0} E\{[Y(t;\varepsilon_2) - Y(t;\varepsilon_1)]^2\} = R_2 - 2R_2 + R_2 = 0 \tag{6.113}$$

Thus, we conclude that $X(t)$ has a m.s. derivative $X'(t)$ if $\partial^2 R_X(t, s)/\partial t\, \partial s$ exists at $s = t$. If $X(t)$ is WSS, then the above conclusion is equivalent to the existence of $\partial^2 R_X(\tau)/\partial^2 \tau$ at $\tau = 0$.

***Note:*** In real analysis, a function $g(\varepsilon)$ of some parameter $\varepsilon$ converges to a finite value if

$$\lim_{\varepsilon_1,\,\varepsilon_2 \to 0} [g(\varepsilon_2) - g(\varepsilon_1)] = 0$$

This is known as the *Cauchy criterion.*

**6.7.** Suppose a random process $X(t)$ has a m.s. derivative $X'(t)$.

(*a*) Find $E[X'(t)]$.

(*b*) Find the cross-correlation function of $X(t)$ and $X'(t)$.

(*c*) Find the autocorrelation function of $X'(t)$.

(*a*) We have

$$\begin{aligned} E[X'(t)] &= E\left[\text{l.i.m.}_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right] \\ &= \lim_{\varepsilon \to 0} E\left[\frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right] \\ &= \lim_{\varepsilon \to 0} \frac{\mu_X(t+\varepsilon) - \mu_X(t)}{\varepsilon} = \mu_X'(t) \end{aligned} \tag{6.114}$$

(*b*) From Eq. (*6.17*), the cross-correlation function of $X(t)$ and $X'(t)$ is

$$R_{XX'}(t, s) = E[X(t)X'(s)] = E\left[X(t) \text{ l.i.m.}_{\varepsilon \to 0} \frac{X(s+\varepsilon) - X(s)}{\varepsilon}\right]$$
$$= \lim_{\varepsilon \to 0} \frac{E[X(t)X(s+\varepsilon)] - E[X(t)X(s)]}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \frac{R_X(t, s+\varepsilon) - R_X(t, s)}{\varepsilon} = \frac{\partial R_X(t, s)}{\partial s} \tag{6.115}$$

(*c*) Using Eq. (*6.115*), the autocorrelation function of $X'(t)$ is

$$R_{X'}(t, s) = E[X'(t)X'(s)] = E\left\{\left[\text{l.i.m.}_{\varepsilon \to 0} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right] X'(s)\right\}$$
$$= \lim_{\varepsilon \to 0} \frac{E[X(t+\varepsilon)X'(s)] - E[X(t)X'(s)]}{\varepsilon}$$
$$= \lim_{\varepsilon \to 0} \frac{R_{XX'}(t+\varepsilon, s) - R_{XX'}(t, s)}{\varepsilon}$$
$$= \frac{\partial R_{XX'}(t, s)}{\partial t} = \frac{\partial^2 R_X(t, s)}{\partial t\, \partial s} \tag{6.116}$$

**6.8.** If $X(t)$ is a WSS random process and has a m.s. derivative $X'(t)$, then show that

(*a*) $$R_{XX'}(\tau) = \frac{d}{d\tau} R_X(\tau) \tag{6.117}$$

(*b*) $$R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau) \tag{6.118}$$

(*a*) For a WSS process $X(t)$, $R_X(t, s) = R_X(s - t)$. Thus, setting $s - t = \tau$ in Eq. (*6.115*) of Prob. 6.7, we obtain $\partial R_X(s-t)/\partial s = dR_X(\tau)/d\tau$ and

$$R_{XX'}(t, t+\tau) = R_{XX'}(\tau) = \frac{dR_X(\tau)}{d\tau}$$

(*b*) Now $\partial R_X(s-t)/\partial t = -dR_X(\tau)/d\tau$. Thus, $\partial^2 R_X(s-t)/\partial t\, \partial s = -d^2 R_X(\tau)/d\tau^2$, and by Eq. (*6.116*) of Prob. 6.7, we have

$$R_{X'}(t, t+\tau) = R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau)$$

**6.9.** Show that the Wiener process $X(t)$ does not have a m.s. derivative.

From Eq. (*5.64*), the autocorrelation function of the Wiener process $X(t)$ is given by

$$R_X(t, s) = \sigma^2 \min(t, s) = \begin{cases} \sigma^2 s & t > s \\ \sigma^2 t & t < s \end{cases}$$

Thus
$$\frac{\partial}{\partial s} R_X(t, s) = \sigma^2 u(t - s) = \begin{cases} \sigma^2 & t > s \\ 0 & t < s \end{cases} \tag{6.119}$$

where $u(t - s)$ is a unit step function defined by

$$u(t - s) = \begin{cases} 1 & t > s \\ 0 & t < s \end{cases}$$

and it is not continuous at $s = t$ (Fig. 6-2). Thus $\partial^2 R_X(t, s)/\partial t\, \partial s$ does not exist at $s = t$, and the Wiener process $X(t)$ does not have a m.s. derivative.

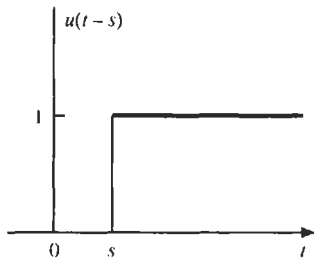



**Fig. 6-2** Shifted unit step function.

Note that although a m.s. derivative does not exist for the Wiener process, we can define a generalized derivative of the Wiener process (see Prob. 6.20).

**6.10.** Show that if $X(t)$ is a normal random process for which the m.s. derivative $X'(t)$ exists, then $X'(t)$ is also a normal random process.

Let $X(t)$ be a normal random process. Now consider

$$Y_\varepsilon(t) = \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$$

Then, $n$ r.v.'s $Y_\varepsilon(t_1)$, $Y_\varepsilon(t_2)$, ..., $Y_\varepsilon(t_n)$ are given by a linear transformation of the jointly normal r.v.'s $X(t_1)$, $X(t_1+\varepsilon)$, $X(t_2)$, $X(t_2+\varepsilon)$, ..., $X(t_n)$, $X(t_n+\varepsilon)$. It then follows by the result of Prob. 5.60 that $Y_\varepsilon(t_1)$, $Y_\varepsilon(t_2)$, ..., $Y_\varepsilon(t_n)$ are jointly normal r.v.'s, and hence $Y_\varepsilon(t)$ is a normal random process. Thus, we conclude that the m.s. derivative $X'(t)$, which is the limit of $Y_\varepsilon(t)$ as $\varepsilon \to 0$, is also a normal random process, since m.s. convergence implies convergence in probability (see Prob. 6.5).

**6.11.** Show that the m.s. integral of a random process $X(t)$ exists if the following integral exists:

$$\int_{t_0}^{t}\int_{t_0}^{t} R_X(\alpha, \beta)\, d\alpha\, d\beta$$

A m.s. integral of $X(t)$ is defined by [Eq. (6.8)]

$$Y(t) = \int_{t_0}^{t} X(\alpha)\, d\alpha = \text{l.i.m.}_{\Delta t_i \to 0} \sum_i X(t_i)\, \Delta t_i$$

Again using the Cauchy criterion, the m.s. integral $Y(t)$ of $X(t)$ exists if

$$\lim_{\Delta t_i,\, \Delta t_k \to 0} E\left\{\left[\sum_i X(t_i)\, \Delta t_i - \sum_k X(t_k)\, \Delta t_k\right]^2\right\} = 0 \tag{6.120}$$

As in the case of the m.s. derivative [Eq. (6.111)], expanding the square, we obtain

$$\begin{aligned} &E\left\{\left[\sum_i X(t_i)\, \Delta t_i - \sum_k X(t_k)\, \Delta t_k\right]^2\right\} \\ &\quad = E\left[\sum_i \sum_k X(t_i)X(t_k)\, \Delta t_i\, \Delta t_k + \sum_i \sum_k X(t_i)X(t_k)\, \Delta t_i\, \Delta t_k - 2\sum_i \sum_k X(t_i)X(t_k)\, \Delta t_i\, \Delta t_k\right] \\ &\quad = \sum_i \sum_k R_X(t_i, t_k)\, \Delta t_i\, \Delta t_k + \sum_i \sum_k R_X(t_i, t_k)\, \Delta t_i\, \Delta t_k - 2\sum_i \sum_k R_X(t_i, t_k)\, \Delta t_i\, \Delta t_k \end{aligned}$$

and Eq. (6.120) holds if

$$\lim_{\Delta t_i,\, \Delta t_k \to 0} \sum_i \sum_k R_X(t_i, t_k)\, \Delta t_i\, \Delta t_k$$

exists, or, equivalently,

$$\int_{t_0}^{t}\int_{t_0}^{t} R_X(\alpha, \beta)\, d\alpha\, d\beta$$

exists.

**6.12.** Let $X(t)$ be the Wiener process with parameter $\sigma^2$. Let

$$Y(t) = \int_0^t X(\alpha)\, d\alpha$$

(*a*) Find the mean and the variance of $Y(t)$.

(*b*) Find the autocorrelation function of $Y(t)$.

(*a*) By assumption 3 of the Wiener process (Sec. 5.7), that is, $E[X(t)] = 0$, we have

$$E[Y(t)] = E\left[\int_0^t X(\alpha)\, d\alpha\right] = \int_0^t E[X(\alpha)]\, d\alpha = 0 \tag{6.121}$$

Then

$$\begin{aligned}\mathrm{Var}[Y(t)] = E[Y^2(t)] &= \int_0^t\int_0^t E[X(\alpha)X(\beta)]\, d\alpha\, d\beta \\ &= \int_0^t\int_0^t R_X(\alpha, \beta)\, d\alpha\, d\beta\end{aligned}$$

By Eq. (*5.64*), $R_X(\alpha, \beta) = \sigma^2 \min(\alpha, \beta)$; thus, referring to Fig. 6-3, we obtain

$$\begin{aligned}\mathrm{Var}[Y(t)] &= \sigma^2 \int_0^t\int_0^t \min(\alpha, \beta)\, d\alpha\, d\beta \\ &= \sigma^2 \int_0^t d\beta \int_0^\beta \alpha\, d\alpha + \sigma^2 \int_0^t d\alpha \int_0^\alpha \beta\, d\beta = \frac{\sigma^2 t^3}{3}\end{aligned} \tag{6.122}$$

(*b*) Let $t > s \geq 0$ and write

$$\begin{aligned}Y(t) &= \int_0^s X(\alpha)\, d\alpha + \int_s^t [X(\alpha) - X(s)]\, d\alpha + (t - s)X(s) \\ &= Y(s) + \int_s^t [X(\alpha) - X(s)]\, d\alpha + (t - s)X(s)\end{aligned}$$

Then, for $t > s \geq 0$,

$$\begin{aligned}R_Y(t, s) &= E[Y(t)Y(s)] \\ &= E[Y^2(s)] + \int_s^t E\{[X(\alpha) - X(s)]Y(s)\}\, d\alpha + (t - s)E[X(s)Y(s)]\end{aligned} \tag{6.123}$$

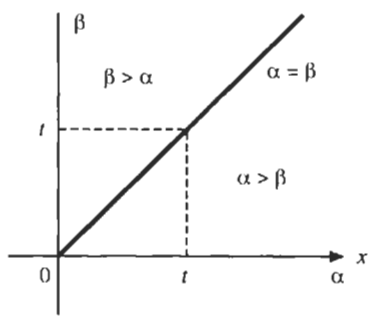



**Fig. 6-3**

Now by Eq. (6.122),

$$E[Y^2(s)] = \text{Var}[Y(s)] = \frac{\sigma^2 s^3}{3}$$

Using assumptions 1, 3, and 4 of the Wiener process (Sec. 5.7), and since $s \le \alpha \le t$, we have

$$\begin{aligned}\int_s^t E\{[X(\alpha) - X(s)]Y(s)\}\, d\alpha &= \int_s^t E\left\{[X(\alpha) - X(s)] \int_0^s X(\beta)\, d\beta\right\} d\alpha \\ &= \int_s^t \int_0^s E\{[X(\alpha) - X(s)][X(\beta) - X(0)]\}\, d\beta\, d\alpha \\ &= \int_s^t \int_0^s E[X(\alpha) - X(s)]E[X(s) - X(0)]\, d\beta\, d\alpha = 0\end{aligned}$$

Finally, for $0 \le \beta \le s$,

$$\begin{aligned}(t - s)E[X(s)Y(s)] &= (t - s) \int_0^s E[X(s)X(\beta)]\, d\beta \\ &= (t - s) \int_0^s R_X(s, \beta)\, d\beta = (t - s) \int_0^s \sigma^2 \min(s, \beta)\, d\beta \\ &= \sigma^2(t - s) \int_0^s \beta\, d\beta = \sigma^2(t - s)\frac{s^2}{2}\end{aligned}$$

Substituting these results into Eq. (6.123), we get

$$R_Y(t, s) = \frac{\sigma^2 s^3}{3} + \sigma^2(t - s)\frac{s^2}{2} = \frac{1}{6}\sigma^2 s^2(3t - s)$$

Since $R_Y(t, s) = R_Y(s, t)$, we obtain

$$R_Y(t, s) = \begin{cases} \frac{1}{6}\sigma^2 s^2(3t - s) & t > s \ge 0 \\ \frac{1}{6}\sigma^2 t^2(3s - t) & s > t \ge 0 \end{cases} \tag{6.124}$$

## POWER SPECTRAL DENSITY

**6.13.** Verify Eqs. (6.13) and (6.14).

From Eq. (6.12),

$$R_X(\tau) = E[X(t)X(t + \tau)]$$

Setting $t + \tau = s$, we get

$$R_X(\tau) = E[X(s - \tau)X(s)] = E[X(s)X(s - \tau)] = R_X(-\tau)$$

Next, we have

$$E\{[X(t) \pm X(t + \tau)]^2\} \ge 0$$

Expanding the square, we have

$$E[X^2(t) \pm 2X(t)X(t + \tau) + X^2(r + \tau)] \ge 0$$

or

$$E[X^2(t)] \pm 2E[X(t)X(t + \tau)] + E[X^2(t + \tau)] \ge 0$$

Thus

$$2R_X(0) \pm 2R_X(\tau) \ge 0$$

from which we obtain Eq. (6.14); that is,

$$R_X(0) \ge |R_X(\tau)|$$

**6.14.** Verify Eqs. (6.18) to (6.20).

By Eq. (*6.17*),

$$R_{XY}(-\tau) = E[X(t)Y(t-\tau)]$$

Setting $t - \tau = s$, we get

$$R_{XY}(-\tau) = E[X(s+\tau)Y(s)] = E[Y(s)X(s+\tau)] = R_{YX}(\tau)$$

Next, from the Cauchy-Schwarz inequality, Eq. (*3.97*) (Prob. 3.35), it follows that

$$\{E[X(t)Y(t+\tau)]\}^2 \le E[X^2(t)]E[Y^2(t+\tau)]$$

or

$$[R_{XY}(\tau)]^2 \le R_X(0)R_Y(0)$$

from which we obtain Eq. (*6.19*); that is,

$$|R_{XY}(\tau)| \le \sqrt{R_X(0)R_Y(0)}$$

Now

$$E\{[X(t) - Y(t+\tau)]^2\} \ge 0$$

Expanding the square, we have

$$E[X^2(t) - 2X(t)Y(t+\tau) + Y^2(t+\tau)] \ge 0$$

or

$$E[X^2(t)] - 2E[X(t)Y(t+\tau)] + E[Y^2(t+\tau)] \ge 0$$

Thus

$$R_X(0) - 2R_{XY}(\tau) + R_Y(0) \ge 0$$

from which we obtain Eq. (*6.20*); that is,

$$R_{XY}(\tau) \le \tfrac{1}{2}[R_X(0) + R_Y(0)]$$

**6.15.** Two random processes $X(t)$ and $Y(t)$ are given by

$$X(t) = A\cos(\omega t + \Theta) \qquad Y(t) = A\sin(\omega t + \Theta)$$

where $A$ and $\omega$ are constants and $\Theta$ is a uniform r.v. over $(0, 2\pi)$. Find the cross-correlation function of $X(t)$ and $Y(t)$ and verify Eq. (*6.18*).

From Eq. (*6.17*), the cross-correlation function of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t, t+\tau) &= E[X(t)Y(t+\tau)] \\ &= E\{A^2\cos(\omega t + \Theta)\sin[\omega(t+\tau) + \Theta]\} \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) - \sin(-\omega\tau)] \\ &= \frac{A^2}{2}\sin\omega\tau = R_{XY}(\tau) \end{aligned} \tag{6.125}$$

Similarly,

$$\begin{aligned} R_{YX}(t, t+\tau) &= E[Y(t)X(t+\tau)] \\ &= E\{A^2\sin(\omega t + \Theta)\cos[\omega(t+\tau) + \Theta]\} \\ &= \frac{A^2}{2}E[\sin(2\omega t + \omega\tau + 2\Theta) + \sin(-\omega\tau)] \\ &= -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau) \end{aligned} \tag{6.126}$$

From Eqs. (*6.125*) and (*6.126*), we see that

$$R_{XY}(-\tau) = \frac{A^2}{2}\sin\omega(-\tau) = -\frac{A^2}{2}\sin\omega\tau = R_{YX}(\tau)$$

which verifies Eq. (*6.18*).

**6.16.** Show that the power spectrum of a (real) random process $X(t)$ is real and verify Eq. (*6.26*).

From Eq. (*6.23*) and expanding the exponential, we have

$$\begin{aligned} S_X(\omega) &= \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}\,d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)(\cos\omega\tau - j\sin\omega\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau\,d\tau - j\int_{-\infty}^{\infty} R_X(\tau)\sin\omega\tau\,d\tau \end{aligned} \tag{6.127}$$

Since $R_X(-\tau) = R_X(\tau)$, $R_X(\tau)\cos\omega\tau$ is an even function of $\tau$ and $R_X(\tau)\sin\omega\tau$ is an odd function of $\tau$, and hence the imaginary term in Eq. (*6.127*) vanishes and we obtain

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)\cos\omega\tau\,d\tau \tag{6.128}$$

which indicates that $S_X(\omega)$ is real. Since $\cos(-\omega\tau) = \cos(\omega\tau)$, it follows that

$$S_X(-\omega) = S_X(\omega)$$

which indicates that the power spectrum of a real random process $X(t)$ is an even function of frequency.

**6.17.** Consider the random process

$$Y(t) = (-1)^{X(t)}$$

where $X(t)$ is a Poisson process with rate $\lambda$. Thus $Y(t)$ starts at $Y(0) = 1$ and switches back and forth from $+1$ to $-1$ at random Poisson times $T_i$, as shown in Fig. 6-4. The process $Y(t)$ is known as the *semirandom telegraph signal* because its initial value $Y(0) = 1$ is not random.

(*a*) Find the mean of $Y(t)$.

(*b*) Find the autocorrelation function of $Y(t)$.

(*a*) We have

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even} \\ -1 & \text{if } X(t) \text{ is odd} \end{cases}$$

Thus, using Eq. (*5.55*), we have

$$\begin{aligned} P[Y(t) = 1] &= P[X(t) = \text{even integer}] \\ &= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \cdots\right] = e^{-\lambda t}\cosh\lambda t \\ P[Y(t) = -1] &= P[X(t) = \text{odd integer}] \\ &= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \cdots\right] = e^{-\lambda t}\sinh\lambda t \end{aligned}$$

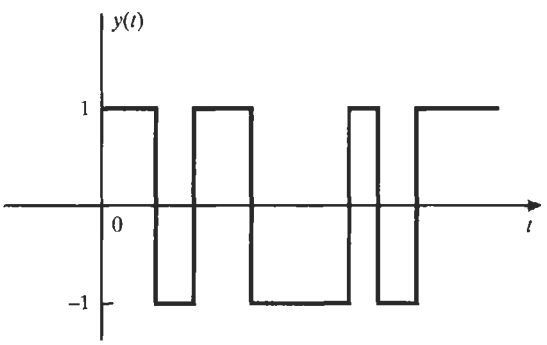



**Fig. 6-4** Semirandom telegraph signal.

Hence
$$\mu_Y(t) = E[Y(t)] = (1)P[Y(t) = 1] + (-1)P[Y(t) = -1]$$
$$= e^{-\lambda t}(\cosh \lambda t - \sinh \lambda t) = e^{-2\lambda t} \tag{6.129}$$

(*b*) Similarly, since $Y(t)Y(t+\tau) = 1$ if there are an even number of events in $(t, t+\tau)$ for $\tau > 0$ and $Y(t)Y(t+\tau) = -1$ if there are an odd number of events, then for $t > 0$ and $t + \tau > 0$,

$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)]$$
$$= (1) \sum_{n \text{ even}} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!} + (-1) \sum_{n \text{ odd}} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$$
$$= e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(-\lambda\tau)^n}{n!} = e^{-\lambda\tau} e^{-\lambda\tau} = e^{-2\lambda\tau}$$

which indicates that $R_Y(t, t+\tau) = R_Y(\tau)$, and by Eq. (*6.13*),

$$R_Y(\tau) = e^{-2\lambda|\tau|} \tag{6.130}$$

Note that since $E[Y(t)]$ is not a constant, $Y(t)$ is not WSS.

**6.18.** Consider the random process

$$Z(t) = A\,Y(t)$$

where $Y(t)$ is the semirandom telegraph signal of Prob. 6.17 and $A$ is a r.v. independent of $Y(t)$ and takes on the values $\pm 1$ with equal probability. The process $Z(t)$ is known as the *random telegraph signal.*

(*a*) Show that $Z(t)$ is WSS.

(*b*) Find the power spectral density of $Z(t)$.

(*a*) Since $E(A) = 0$ and $E(A^2) = 1$, the mean of $Z(t)$ is

$$\mu_Z(t) = E[Z(t)] = E(A)E[Y(t)] = 0 \tag{6.131}$$

and the autocorrelation of $Z(t)$ is

$$R_Z(t, t+\tau) = E[A^2 Y(t)Y(t+\tau)] = E(A^2)E[Y(t)Y(t+\tau)] = R_Y(t, t+\tau)$$

Thus, using Eq. (*6.130*), we obtain

$$R_Z(t, t+\tau) = R_Z(\tau) = e^{-2\lambda|\tau|} \tag{6.132}$$

Thus, we see that $Z(t)$ is WSS.

(*b*) Taking the Fourier transform of Eq. (*6.132*) (see Appendix B), we see that the power spectrum of $Z(t)$ is given by

$$S_Z(\omega) = \frac{4\lambda}{\omega^2 + 4\lambda^2} \tag{6.133}$$

**6.19.** Let $X(t)$ and $Y(t)$ be both zero-mean and WSS random processes. Consider the random process $Z(t)$ defined by

$$Z(t) = X(t) + Y(t)$$

(*a*) Determine the autocorrelation function and the power spectral density of $Z(t)$, (i) if $X(t)$ and $Y(t)$ are jointly WSS; (ii) if $X(t)$ and $Y(t)$ are orthogonal.

(*b*) Show that if $X(t)$ and $Y(t)$ are orthogonal, then the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.

(*a*) The autocorrelation of $Z(t)$ is given by

$$R_Z(t, s) = E[Z(t)Z(s)] = E\{[X(t) + Y(t)][X(s) + Y(s)]\}$$
$$= E[X(t)X(s)] + E[X(t)Y(s)] + E[Y(t)X(s)] + E[Y(t)Y(s)]$$
$$= R_X(t, s) + R_{XY}(t, s) + R_{YX}(t, s) + R_Y(t, s)$$

(i) If $X(t)$ and $Y(t)$ are jointly WSS, then we have

$$R_Z(\tau) = R_X(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_Y(\tau)$$

where $\tau = s - t$. Taking the Fourier transform of the above expression, we obtain

$$S_Z(\omega) = S_X(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_Y(\omega)$$

(ii) If $X(t)$ and $Y(t)$ are orthogonal [Eq. (*6.21*)],

$$R_{XY}(\tau) = R_{YX}(\tau) = 0$$

Then

$$R_Z(\tau) = R_X(\tau) + R_Y(\tau) \tag{6.134a}$$

$$S_Z(\omega) = S_X(\omega) + S_Y(\omega) \tag{6.134b}$$

(*b*) Setting $\tau = 0$ in Eq. (*6.134a*), and using Eq. (*6.15*), we get

$$E[Z^2(t)] = E[X^2(t)] + E[Y^2(t)]$$

which indicates that the mean square of $Z(t)$ is equal to the sum of the mean squares of $X(t)$ and $Y(t)$.

## WHITE NOISE

**6.20.** Using the notion of generalized derivative, show that the generalized derivative $X'(t)$ of the Wiener process $X(t)$ is a white noise.

From Eq. (*5.64*),

$$R_X(t, s) = \sigma^2 \min(t, s)$$

and from Eq. (*6.119*) (Prob. 6.9), we have

$$\frac{\partial}{\partial s} R_X(t, s) = \sigma^2 u(t - s) \tag{6.135}$$

Now, using the $\delta$ function, the generalized derivative of a unit step function $u(t)$ is given by

$$\frac{d}{dt} u(t) = \delta(t)$$

Applying the above relation to Eq. (*6.135*), we obtain

$$\frac{\partial^2}{\partial t\, \partial s} R_X(t, s) = \sigma^2 \frac{\partial}{\partial t} u(t - s) = \sigma^2 \delta(t - s) \tag{6.136}$$

which is, by Eq. (*6.116*) (Prob. 6.7), the autocorrelation function of the generalized derivative $X'(t)$ of the Wiener process $X(t)$; that is,

$$R_{X'}(t, s) = \sigma^2 \delta(t - s) = \sigma^2 \delta(\tau) \tag{6.137}$$

where $\tau = t - s$. Thus, by definition (*6.43*), we see that the generalized derivative $X'(t)$ of the Wiener process $X(t)$ is a white noise.

Recall that the Wiener process is a normal process and its derivative is also normal (see Prob. 6.10). Hence, the generalized derivative $X'(t)$ of the Wiener process is called *white normal* (or *white gaussian*) *noise*.

**6.21.** Let $X(t)$ be a Poisson process with rate $\lambda$. Let

$$Y(t) = X(t) - \lambda t$$

Show that the generalized derivative $Y'(t)$ of $Y(t)$ is a white noise.

Since $Y(t) = X(t) - \lambda t$, we have formally

$$Y'(t) = X'(t) - \lambda \tag{6.138}$$

Then
$$E[Y'(t)] = E[X'(t) - \lambda] = E[X'(t)] - \lambda \tag{6.139}$$

$$\begin{aligned} R_{Y'}(t, s) &= E[Y'(t)Y'(s)] = E\{[X'(t) - \lambda][X'(s) - \lambda]\} \\ &= E[X'(t)X'(s) - \lambda X'(s) - \lambda X'(t) + \lambda^2] \\ &= E[X'(t)X'(s)] - \lambda E[X'(s)] - \lambda E[X'(t)] + \lambda^2 \end{aligned} \tag{6.140}$$

Now, from Eqs. (5.56) and (5.60), we have

$$E[X(t)] = \lambda t$$

$$R_X(t, s) = \lambda \min(t, s) + \lambda^2 ts$$

Thus
$$E[X'(t)] = \lambda \qquad \text{and} \qquad E[X'(s)] = \lambda \tag{6.141}$$

and from Eqs. (6.7) and (6.137),

$$E[X'(t)X'(s)] = R_{X'}(t, s) = \frac{\partial^2 R_X(t, s)}{\partial t\, \partial s} = \lambda\delta(t - s) + \lambda^2 \tag{6.142}$$

Substituting Eq. (6.141) into Eq. (6.139), we obtain

$$E[Y'(t)] = 0 \tag{6.143}$$

Substituting Eqs. (6.141) and (6.142) into Eq. (6.140), we get

$$R_{Y'}(t, s) = \lambda\delta(t - s) \tag{6.144}$$

Hence we see that $Y'(t)$ is a zero-mean WSS random process, and by definition (6.43), $Y'(t)$ is a white noise with $\sigma^2 = \lambda$. The process $Y'(t)$ is known as the *Poisson white noise.*

**6.22.** Let $X(t)$ be a white normal noise. Let

$$Y(t) = \int_0^t X(\alpha)\, d\alpha$$

(*a*) Find the autocorrelation function of $Y(t)$.

(*b*) Show that $Y(t)$ is the Wiener process.

(*a*) From Eq. (6.137) of Prob. 6.20,

$$R_X(t, s) = \sigma^2\delta(t - s)$$

Thus, by Eq. (6.11), the autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_Y(t, s) &= \int_0^t \int_0^s R_X(\alpha, \beta)\, d\beta\, d\alpha \\ &= \int_0^s \int_0^t \sigma^2\delta(\alpha - \beta)\, d\alpha\, d\beta \\ &= \sigma^2 \int_0^s u(t - \beta)\, d\beta \\ &= \sigma^2 \int_0^{\min(t, s)} d\beta = \sigma^2 \min(t, s) \end{aligned} \tag{6.145}$$

(*b*) Comparing Eq. (6.145) and Eq. (5.64), we see that $Y(t)$ has the same autocorrelation function as the Wiener process. In addition, $Y(t)$ is normal, since $X(t)$ is a normal process and $Y(0) = 0$. Thus, we conclude that $Y(t)$ is the Wiener process.

**6.23.** Let $Y(n) = X(n) + W(n)$, where $X(n) = A$ (for all $n$) and $A$ is a r.v. with zero mean and variance $\sigma_A^2$, and $W(n)$ is a discrete-time white noise with average power $\sigma^2$. It is also assumed that $X(n)$ and $W(n)$ are independent.

(*a*) Show that $Y(n)$ is WSS.

(*b*) Find the power spectral density $S_Y(\Omega)$ of $Y(n)$.

(*a*) The mean of $Y(n)$ is

$$E[Y(n)] = E[X(n)] + E[W(n)] = E(A) + E[W(n)] = 0$$

The autocorrelation function of $Y(n)$ is

$$\begin{aligned} R_Y(n, n+k) &= E\{[X(n) + W(n)][X(n+k) + W(n+k)]\} \\ &= E[X(n)X(n+k)] + E[X(n)]E[W(n+k)] + E[W(n)]E[X(n+k)] + E[W(n)W(n+k)] \\ &= E(A^2) + R_W(k) = \sigma_A{}^2 + \sigma^2\delta(k) = R_Y(k) \end{aligned} \tag{6.146}$$

Thus $Y(n)$ is WSS.

(*b*) Taking the Fourier transform of Eq. (*6.146*), we obtain

$$S_Y(\Omega) = 2\pi\sigma_A{}^2\delta(\Omega) + \sigma^2 \qquad -\pi < \Omega < \pi \tag{6.147}$$

## RESPONSE OF LINEAR SYSTEMS TO RANDOM INPUTS

**6.24.** Derive Eq. (*6.58*).

Using Eq. (*6.56*), we have

$$\begin{aligned} R_Y(t, s) &= E[Y(t)Y(s)] \\ &= E\left[\int_{-\infty}^{\infty} h(\alpha)X(t-\alpha)\,d\alpha \int_{-\infty}^{\infty} h(\beta)X(s-\beta)\,d\beta\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)E[X(t-\alpha)X(s-\beta)]\,d\alpha\,d\beta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(t-\alpha, s-\beta)\,d\alpha\,d\beta \end{aligned}$$

**6.25.** Derive Eq. (*6.63*).

From Eq. (*6.62*), we have

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau + \alpha - \beta)\,d\alpha\,d\beta$$

Taking the Fourier transform of $R_Y(\tau)$, we obtain

$$S_Y(\omega) = \int_{-\infty}^{\infty} R_Y(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\tau + \alpha - \beta)e^{-j\omega\tau}\,d\alpha\,d\beta\,d\tau$$

Letting $\tau + \alpha - \beta = \lambda$, we get

$$\begin{aligned} S_Y(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)h(\beta)R_X(\lambda)e^{-j\omega(\lambda-\alpha+\beta)}\,d\alpha\,d\beta\,d\lambda \\ &= \int_{-\infty}^{\infty} h(\alpha)e^{j\omega\alpha}\,d\alpha \int_{-\infty}^{\infty} h(\beta)e^{-j\omega\beta}\,d\beta \int_{-\infty}^{\infty} R_X(\lambda)e^{-j\omega\lambda}\,d\lambda \\ &= H(-\omega)H(\omega)S_X(\omega) \\ &= H^*(\omega)H(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega) \end{aligned}$$

**6.26.** A WSS random process $X(t)$ with autocorrelation function

$$R_X(\tau) = e^{-a|\tau|}$$

where $a$ is a real positive constant, is applied to the input of an LTI system with impulse response

$$h(t) = e^{-bt}u(t)$$

where $b$ is a real positive constant. Find the autocorrelation function of the output $Y(t)$ of the system.

The frequency response $H(\omega)$ of the system is

$$H(\omega) = \mathcal{F}[h(t)] = \frac{1}{j\omega + b}$$

The power spectral density of $X(t)$ is

$$S_X(\omega) = \mathcal{F}[R_X(\tau)] = \frac{2a}{\omega^2 + a^2}$$

By Eq. (*6.63*), the power spectral density of $Y(t)$ is

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \left(\frac{1}{\omega^2 + b^2}\right)\left(\frac{2a}{\omega^2 + a^2}\right)$$
$$= \frac{a}{(a^2 - b^2)b}\left(\frac{2b}{\omega^2 + b^2}\right) - \frac{b}{(a^2 - b^2)b}\left(\frac{2a}{\omega^2 + a^2}\right)$$

Taking the inverse Fourier transform of both sides of the above equation, we obtain

$$R_Y(\tau) = \frac{1}{(a^2 - b^2)b}(ae^{-b|\tau|} - be^{-a|\tau|})$$

**6.27.** Verify Eq. (*6.25*), that is, the power spectral density of any WSS process $X(t)$ is real and $S_X(\omega) \geq 0$.

The realness of $S_X(\omega)$ was shown in Prob. 6.16. Consider an ideal bandpass filter with frequency response (Fig. 6-5)

$$H(\omega) = \begin{cases} 1 & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

with a random process $X(t)$ as its input.

From Eq. (*6.63*), it follows that the power spectral density $S_Y(\omega)$ of the output $Y(t)$ equals

$$S_Y(\omega) = \begin{cases} S_X(\omega) & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence, from Eq. (*6.27*), we have

$$E[Y^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Y(\omega)\, d\omega = 2\frac{1}{2\pi}\int_{\omega_1}^{\omega_2} S_X(\omega)\, d\omega \geq 0$$

which indicates that the area of $S_X(\omega)$ in any interval of $\omega$ is nonnegative. This is possible only if $S_X(\omega) \geq 0$ for every $\omega$.

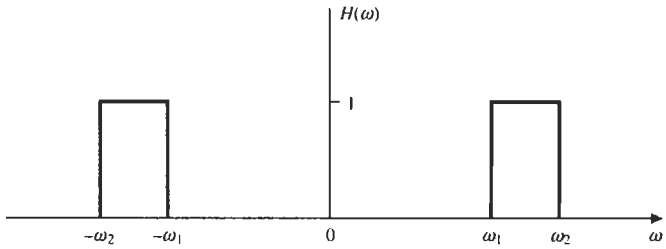



**Fig. 6-5**

**6.28.** Verify Eq. (6.72).

From Eq. (6.71), we have

$$R_Y(k) = \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(k + i - l)$$

Taking the Fourier transform of $R_Y(k)$, we obtain

$$S_Y(\Omega) = \sum_{k=-\infty}^{\infty} R_Y(k)e^{-j\Omega k} = \sum_{k=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(k + i - l)e^{-j\Omega k}$$

Letting $k + i - l = n$, we get

$$\begin{aligned} S_Y(\Omega) &= \sum_{n=-\infty}^{\infty} \sum_{i=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} h(i)h(l)R_X(n)e^{-j\Omega(n-i+l)} \\ &= \sum_{i=-\infty}^{\infty} h(i)e^{j\Omega i} \sum_{l=-\infty}^{\infty} h(l)e^{-j\Omega l} \sum_{n=-\infty}^{\infty} R_X(n)e^{-j\Omega n} \\ &= H(-\Omega)H(\Omega)S_X(\Omega) \\ &= H^*(\Omega)H(\Omega)S_X(\Omega) = |H(\Omega)|^2 S_X(\Omega) \end{aligned}$$

**6.29.** The discrete-time system shown in Fig. 6-6 consists of one unit delay element and one scalar multiplier ($a < 1$). The input $X(n)$ is discrete-time white noise with average power $\sigma^2$. Find the spectral density and average power of the output $Y(n)$.

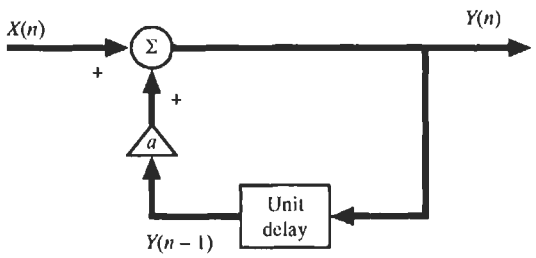



**Fig. 6-6**

From Fig. 6-6, $Y(n)$ and $X(n)$ are related by

$$Y(n) = aY(n - 1) + X(n) \tag{6.148}$$

The impulse response $h(n)$ of the system is defined by

$$h(n) = ah(n - 1) + \delta(n) \tag{6.149}$$

Solving Eq. (6.149), we obtain

$$h(n) = a^n u(n) \tag{6.150}$$

where $u(n)$ is the unit step sequence defined by

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Taking the Fourier transform of Eq. (6.150), we obtain

$$H(\Omega) = \sum_{n=0}^{\infty} a^n e^{-j\Omega n} = \frac{1}{1 - ae^{-j\Omega}} \qquad a < 1, |\Omega| < \pi$$

Now, by Eq. (6.48),

$$S_X(\Omega) = \sigma^2 \qquad |\Omega| < \pi$$

and by Eq. (6.72), the power spectral density of $Y(n)$ is

$$\begin{aligned} S_Y(\Omega) &= |H(\Omega)|^2 S_X(\Omega) = H(\Omega)H(-\Omega)S_X(\Omega) \\ &= \frac{\sigma^2}{(1 - ae^{-j\Omega})(1 - ae^{j\Omega})} \\ &= \frac{\sigma^2}{1 + a^2 - 2a\cos\Omega} \qquad |\Omega| < \pi \end{aligned} \tag{6.151}$$

Taking the inverse Fourier transform of Eq. (6.151), we obtain

$$R_Y(k) = \frac{\sigma^2}{1 - a^2}\, a^{|k|}$$

Thus, by Eq. (6.33), the average power of $Y(n)$ is

$$E[Y^2(n)] = R_Y(0) = \frac{\sigma^2}{1 - a^2}$$

**6.30.** Let $Y(t)$ be the output of an LTI system with impulse response $h(t)$, when $X(t)$ is applied as input. Show that

(*a*) $$R_{XY}(t, s) = \int_{-\infty}^{\infty} h(\beta)R_X(t, s - \beta)\, d\beta \tag{6.152}$$

(*b*) $$R_Y(t, s) = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t - \alpha, s)\, d\alpha \tag{6.153}$$

(*a*) Using Eq. (6.56), we have

$$\begin{aligned} R_{XY}(t, s) &= E[X(t)Y(s)] = E\left[X(t)\int_{-\infty}^{\infty} h(\beta)X(s - \beta)\, d\beta\right] \\ &= \int_{-\infty}^{\infty} h(\beta)E[X(t)X(s - \beta)]\, d\beta = \int_{-\alpha}^{\infty} h(\beta)R_X(t, s - \beta)\, d\beta \end{aligned}$$

(*b*) Similarly,

$$\begin{aligned} R_Y(t, s) &= E[Y(t)Y(s)] = E\left[\int_{-\infty}^{\infty} h(\alpha)X(t - \alpha)\, d\alpha\ Y(s)\right] \\ &= \int_{-\infty}^{\infty} h(\alpha)E[X(t - \alpha)Y(s)]\, d\alpha = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(t - \alpha, s)\, d\alpha \end{aligned}$$

**6.31.** Let $Y(t)$ be the output of an LTI system with impulse response $h(t)$ when a WSS random process $X(t)$ is applied as input. Show that

(*a*) $S_{XY}(\omega) = H(\omega)S_X(\omega)$ (6.154)

(*b*) $S_Y(\omega) = H^*(\omega)S_{XY}(\omega)$ (6.155)

(*a*) If $X(t)$ is WSS, then Eq. (6.152) of Prob. 6.30 becomes

$$R_{XY}(t, s) = \int_{-\infty}^{\infty} h(\beta)R_X(s - t - \beta)\, d\beta \tag{6.156}$$

which indicates that $R_{XY}(t, s)$ is a function of the time difference $\tau = s - t$ only. Hence

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\beta)R_X(\tau - \beta)\, d\beta \tag{6.157}$$

Taking the Fourier transform of Eq. (*6.157*), we obtain

$$\begin{aligned} S_{XY}(\omega) &= \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\beta)R_X(\tau-\beta)e^{-j\omega\tau}\,d\beta\,d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\beta)R_X(\lambda)e^{-j\omega(\lambda+\beta)}\,d\beta\,d\lambda \\ &= \int_{-\infty}^{\infty} h(\beta)e^{-j\omega\beta}\,d\beta \int_{-\infty}^{\infty} R_X(\lambda)e^{-j\omega\lambda}\,d\lambda = H(\omega)S_X(\omega) \end{aligned}$$

(*b*) Similarly, if $X(t)$ is WSS, then by Eq. (*6.156*), Eq. (*6.153*) becomes

$$R_Y(t, s) = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(s - t + \alpha)\,d\alpha$$

which indicates that $R_Y(t, s)$ is a function of the time difference $\tau = s - t$ only. Hence

$$R_Y(\tau) = \int_{-\infty}^{\infty} h(\alpha)R_{XY}(\tau + \alpha)\,d\alpha \tag{6.158}$$

Taking the Fourier transform $R_Y(\tau)$, we obtain

$$\begin{aligned} S_Y(\omega) &= \int_{-\infty}^{\infty} R_Y(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)R_{XY}(\tau+\alpha)e^{-j\omega\tau}\,d\alpha\,d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\alpha)R_{XY}(\lambda)e^{-j\omega(\lambda-\alpha)}\,d\alpha\,d\lambda \\ &= \int_{-\infty}^{\infty} h(\alpha)e^{j\omega\alpha}\,d\alpha \int_{-\infty}^{\infty} R_{XY}(\lambda)e^{-j\omega\lambda}\,d\lambda \\ &= H(-\omega)S_{XY}(\omega) = H^*(\omega)S_{XY}(\omega) \end{aligned}$$

Note that from Eqs. (*6.154*) and (*6.155*), we obtain Eq. (*6.63*); that is,

$$S_Y(\omega) = H^*(\omega)S_{XY}(\omega) = H^*(\omega)H(\omega)S_X(\omega) = |H(\omega)|^2 S_X(\omega)$$

**6.32.** Consider a WSS process $X(t)$ with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(\omega)$. Let $X'(t) = dX(t)/dt$. Show that

(*a*) $$R_{XX'}(\tau) = \frac{d}{d\tau} R_X(\tau) \tag{6.159}$$

(*b*) $$R_{X'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau) \tag{6.160}$$

(*c*) $$S_{X'}(\omega) = \omega^2 S_X(\omega) \tag{6.161}$$

(*a*) If $X(t)$ is the input to a differentiator, then its output is $Y(t) = X'(t)$. The frequency response of a differentiator is known as $H(\omega) = j\omega$. Then from Eq. (*6.154*),

$$S_{XX'}(\omega) = H(\omega)S_X(\omega) = j\omega S_X(\omega)$$

Taking the inverse Fourier transform of both sides, we obtain

$$R_{XX'}(\tau) = \frac{d}{d\tau} R_X(\tau)$$

(*b*) From Eq. (*6.155*),

$$S_{X'}(\omega) = H^*(\omega)S_{XX'}(\omega) = -j\omega S_{XX'}(\omega)$$

Again taking the inverse Fourier transform of both sides and using the result of part (*a*), we have

$$R_{X'}(\tau) = -\frac{d}{d\tau} R_{XX'}(\tau) = -\frac{d^2}{d\tau^2} R_X(\tau)$$

(*c*) From Eq. (*6.63*),

$$S_{X'}(\omega) = |H(\omega)|^2 S_X(\omega) = |j\omega|^2 S_X(\omega) = \omega^2 S_X(\omega)$$

Note that Eqs. (*6.159*) and (*6.160*) were proved in Prob. 6.8 by a different method.

## FOURIER SERIES AND KARHUNEN-LOÉVE EXPANSIONS

**6.33.** Verify Eqs. (*6.80*) and (*6.81*).

From Eq. (*6.78*),

$$X_n = \frac{1}{T}\int_0^T X(t)e^{-jn\omega_0 t}\,dt \qquad \omega_0 = 2\pi/T$$

Since $X(t)$ is WSS, $E[X(t)] = \mu_X$, and we have

$$E(X_n) = \frac{1}{T}\int_0^T E[X(t)]e^{-jn\omega_0 t}\,dt$$

$$= \mu_X \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\,dt = \mu_X\,\delta(n)$$

Again using Eq. (*6.78*), we have

$$E(X_n X_m^*) = E\left[X_n \frac{1}{T}\int_0^T X^*(s)e^{jm\omega_0 s}\,ds\right]$$

$$= \frac{1}{T}\int_0^T E[X_n X^*(s)]e^{jm\omega_0 s}\,ds$$

Now

$$E[X_n X^*(s)] = E\left[\frac{1}{T}\int_0^T X(t)e^{-jn\omega_0 t}\,dt\; X^*(s)\right]$$

$$= \frac{1}{T}\int_0^T E[X(t)X^*(s)]e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{T}\int_0^T R_X(t-s)e^{-jn\omega_0 t}\,dt$$

Letting $t - s = \tau$, and using Eq. (*6.76*), we obtain

$$E[X_n X^*(s)] = \frac{1}{T}\int_0^T R_X(\tau)e^{-jn\omega_0(\tau+s)}\,d\tau$$

$$= \left\{\frac{1}{T}\int_0^T R_X(\tau)e^{-jn\omega_0\tau}\,d\tau\right\}e^{-jn\omega_0 s} = c_n e^{-jn\omega_0 s} \tag{6.162}$$

Thus

$$E(X_n X_m^*) = \frac{1}{T}\int_0^T c_n e^{-jn\omega_0 s}e^{jm\omega_0 s}\,ds$$

$$= c_n \frac{1}{T}\int_0^T e^{-j(n-m)\omega_0 s}\,ds = c_n\,\delta(n-m)$$

**6.34.** Let $\hat{X}(t)$ be the Fourier series representation of $X(t)$ shown in Eq. (*6.77*). Verify Eq. (*6.79*).

From Eq. (6.77), we have

$$E\{|X(t)-\hat{X}(t)|^2\} = E\left\{\left|X(t)-\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right|^2\right\}$$

$$= E\left\{\left[X(t)-\sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t}\right]\left[X^*(t)-\sum_{n=-\infty}^{\infty} X_n^* e^{-jn\omega_0 t}\right]\right\}$$

$$= E[|X(t)|^2] - \sum_{n=-\infty}^{\infty} E[X_n^* X(t)]e^{-jn\omega_0 t}$$

$$- \sum_{n=-\infty}^{\infty} E[X_n X^*(t)]e^{jn\omega_0 t} + \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[X_n X_m^*]e^{j(n-m)\omega_0 t}$$

Now, by Eqs. (6.81) and (6.162), we have

$$E[X_n^* X(t)] = c_n^* e^{jn\omega_0 t}$$

$$E[X_n X^*(t)] = c_n e^{-jn\omega_0 t}$$

$$E(X_n X_m^*) = c_n \delta(n-m)$$

Using these results, finally we obtain

$$E\{|X(t)-\hat{X}(t)|^2\} = R_X(0) - \sum_{n=-\infty}^{\infty} c_n^* - \sum_{n=-\infty}^{\infty} c_n + \sum_{n=-\infty}^{\infty} c_n = 0$$

since each sum above equals $R_X(0)$ [see Eq. (6.75)].

**6.35.** Let $X(t)$ be m.s. periodic and represented by the Fourier series [Eq. (6.77)]

$$X(t) = \sum_{n=-\infty}^{\infty} X_n e^{jn\omega_0 t} \qquad \omega_0 = 2\pi/T_0$$

Show that

$$E[|X(t)|^2] = \sum_{n=-\infty}^{\infty} E(|X_n|^2) \tag{6.163}$$

From Eq. (6.81), we have

$$E(|X_n|^2) = E(X_n X_n^*) = c_n \tag{6.164}$$

Setting $\tau = 0$ in Eq. (6.75), we obtain

$$E[|X(t)|^2] = R_X(0) = \sum_{n=-\infty}^{\infty} c_n = \sum_{n=-\infty}^{\infty} E(|X_n|^2)$$

Equation (6.163) is known as *Parseval's theorem* for the Fourier series.

**6.36.** If a random process $X(t)$ is represented by a Karhunen-Loéve expansion [Eq. (6.82)]

$$X(t) = \sum_{n=1}^{\infty} X_n \phi_n(t) \qquad 0 < t < T$$

and $X_n$'s are orthogonal, show that $\phi_n(t)$ must satisfy integral equation (6.86); that is,

$$\int_0^T R_X(t, s)\phi_n(s)\, ds = \lambda_n \phi_n(t) \qquad 0 < t < T$$

Consider

$$X(t)X_n^* = \sum_{m=1}^{\infty} X_m X_n^* \phi_m(t)$$

Then
$$E[X(t)X_n^*] = \sum_{m=1}^{\infty} E(X_m X_n^*]\phi_m(t) = E(|X_n|^2)\phi_n(t) \tag{6.165}$$

since $X_n$'s are orthogonal; that is, $E(X_m X_n^*) = 0$ if $m \neq n$. But by Eq. (6.84),

$$\begin{aligned} E[X(t)X_n^*] &= E\left[X(t)\int_0^T X^*(s)\phi_n(s)\,ds\right] \\ &= \int_0^T E[X(t)X^*(s)]\phi_n(s)\,ds \\ &= \int_0^T R_X(t, s)\phi_n(s)\,ds \end{aligned} \tag{6.166}$$

Thus, equating Eqs. (6.165) and (6.166), we obtain

$$\int_0^T R_X(t, s)\phi_n(s)\,ds = E(|X_n|^2)\phi_n(t) = \lambda_n\phi_n(t)$$

where $\lambda_n = E(|X_n|^2)$.

**6.37.** Let $\hat{X}(t)$ be the Karhunen-Loéve expansion of $X(t)$ shown in Eq. (6.82). Verify Eq. (6.88).

From Eqs. (6.166) and (6.86), we have

$$E[X(t)X_m^*] = \int_0^T R_X(t, s)\phi_m(s)\,ds = \lambda_m\phi_m(t) \tag{6.167}$$

Now by Eqs. (6.83), (6.84), and (6.167) we obtain

$$\begin{aligned} E(X_n X_m^*) &= E\left[\int_0^T X(t)\phi_n^*(t)\,dt\ X_m^*\right] = \int_0^T E[X(t)X_m^*]\phi_n^*(t)\,dt \\ &= \int_0^T \lambda_m\phi_m(t)\phi_n^*(t)\,dt = \lambda_m\int_0^T \phi_m(t)\phi_n^*(t)\,dt \\ &= \lambda_m\delta(m-n) = \lambda_n\delta(n-m) \end{aligned} \tag{6.168}$$

**6.38.** Let $\hat{X}(t)$ be the Karhunen-Loéve expansion of $X(t)$ shown in Eq. (6.82). Verify Eq. (6.85).

From Eq. (6.82), we have

$$\begin{aligned} E[|X(t) - \hat{X}(t)|^2] &= E\left\{\left|X(t) - \sum_{n=1}^{\infty} X_n\phi_n(t)\right|^2\right\} \\ &= E\left\{\left[X(t) - \sum_{n=1}^{\infty} X_n\phi_n(t)\right]\left[X^*(t) - \sum_{n=1}^{\infty} X_n^*\phi_n^*(t)\right]\right\} \\ &= E[|X(t)|^2] - \sum_{n=1}^{\infty} E[X(t)X_n^*]\phi_n^*(t) \\ &\quad - \sum_{n=1}^{\infty} E[X^*(t)X_n]\phi_n(t) + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} E(X_n X_m^*)\phi_n(t)\phi_m^*(t) \end{aligned}$$

Using Eqs. (6.167) and (6.168), we have

$$\begin{aligned} E[|X(t) - \hat{X}(t)|^2] &= R_X(t, t) - \sum_{n=1}^{\infty}\lambda_n\phi_n(t)\phi_n^*(t) - \sum_{n=1}^{\infty}\lambda_n^*\phi_n(t)\phi_n^*(t) + \sum_{n=1}^{\infty}\lambda_n\phi_n(t)\phi_n^*(t) \\ &= 0 \end{aligned}$$

since by Mercer's theorem [Eq. (6.87)]

$$R_X(t, t) = \sum_{n=1}^{\infty}\lambda_n\phi_n(t)\phi_n^*(t)$$

and $$\lambda_n = E(|X_n|^2) = \lambda_n^*$$

**6.39.** Find the Karhunen-Loéve expansion of the Wiener process $X(t)$.

From Eq. (*5.64*),

$$R_X(t, s) = \sigma^2 \min(t, s) = \begin{cases} \sigma^2 s & s < t \\ \sigma^2 t & s > t \end{cases}$$

Substituting the above expression into Eq. (*6.86*), we obtain

$$\sigma^2 \int_0^T \min(t, s)\phi_n(s)\, ds = \lambda_n \phi_n(t) \qquad 0 < t < T \tag{6.169}$$

or

$$\sigma^2 \int_0^t s\phi_n(s)\, ds + \sigma^2 t \int_t^T \phi_n(s)\, ds = \lambda_n \phi_n(t) \tag{6.170}$$

Differentiating Eq. (*6.170*) with respect to $t$, we get

$$\sigma^2 \int_t^T \phi_n(s)\, ds = \lambda_n \phi_n'(t) \tag{6.171}$$

Differentiating Eq. (*6.171*) with respect to $t$ again, we obtain

$$\phi_n''(t) + \frac{\sigma^2}{\lambda_n} \phi_n(t) = 0 \tag{6.172}$$

A general solution of Eq. (*6.172*) is

$$\phi_n(t) = a_n \sin \omega_n t + b_n \cos \omega_n t \qquad \omega_n = \sigma/\sqrt{\lambda_n}$$

In order to determine the values of $a_n$, $b_n$, and $\lambda_n$ (or $\omega_n$), we need appropriate boundary conditions. From Eq. (*6.170*), we see that $\phi_n(0) = 0$. This implies that $b_n = 0$. From Eq. (*6.171*), we see that $\phi_n'(T) = 0$. This implies that

$$\omega_n = \frac{\sigma}{\sqrt{\lambda_n}} = \frac{(2n-1)\pi}{2T} = \frac{(n-\frac{1}{2})\pi}{T} \qquad n = 1, 2, \ldots$$

Therefore the eigenvalues are given by

$$\lambda_n = \frac{\sigma^2 T^2}{(n-\frac{1}{2})^2 \pi^2} \qquad n = 1, 2, \ldots \tag{6.173}$$

The normalization requirement [Eq. (*6.83*)] implies that

$$\int_0^T (a_n \sin \omega_n t)^2\, dt = \frac{a_n^2 T}{2} = 1 \rightarrow a_n = \sqrt{\frac{2}{T}}$$

Thus, the eigenfunctions are given by

$$\phi_n(t) = \sqrt{\frac{2}{T}} \sin\left(n - \frac{1}{2}\right) \frac{\pi}{T} t \qquad 0 < t < T \tag{6.174}$$

and the Karhunen-Loéve expansion of the Wiener process $X(t)$ is

$$X(t) = \sqrt{\frac{2}{T}} \sum_{n=1}^{\infty} X_n \sin\left(n - \frac{1}{2}\right) \frac{\pi}{T} t \qquad 0 < t < T \tag{6.175}$$

where $X_n$ are given by

$$X_n = \sqrt{\frac{2}{T}} \int_0^{\infty} X(t) \sin\left(n - \frac{1}{2}\right) \frac{\pi}{T} t$$

and they are uncorrelated with variance $\lambda_n$.

**6.40.** Find the Karhunen-Loéve expansion of the white normal (or white gaussian) noise $W(t)$.

From Eq. (*6.43*),

$$R_W(t, s) = \sigma^2 \delta(t - s)$$

Substituting the above expression into Eq. (*6.86*), we obtain

$$\sigma^2 \int_0^T \delta(t - s)\phi_n(s)\, ds = \lambda_n \phi_n(t) \qquad 0 < t < T$$

or [by Eq. (*6.44*)]

$$\sigma^2 \phi_n(t) = \lambda_n \phi_n(t) \tag{6.176}$$

which indicates that all $\lambda_n = \sigma^2$ and $\phi_n(t)$ are arbitrary. Thus, any complete orthogonal set $\{\phi_n(t)\}$ with corresponding eigenvalues $\lambda_n = \sigma^2$ can be used in the Karhunen-Loéve expansion of the white gaussian noise.

## FOURIER TRANSFORM OF RANDOM PROCESSES

**6.41.** Derive Eq. (*6.94*).

From Eq. (*6.89*),

$$\tilde{X}(\omega_1) = \int_{-\infty}^{\infty} X(t)e^{-j\omega_1 t}\, dt \qquad \tilde{X}(\omega_2) = \int_{-\infty}^{\infty} X(s)e^{-j\omega_2 s}\, ds$$

Then

$$R_{\tilde{X}}(\omega_1, \omega_2) = E[\tilde{X}(\omega_1)\tilde{X}^*(\omega_2)] = E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t)X^*(s)e^{-j(\omega_1 t - \omega_2 s)}\, dt\, ds\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t)X^*(s)]e^{-j(\omega_1 t - \omega_2 s)}\, dt\, ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t, s)e^{-j(\omega_1 t + (-\omega_2)s)}\, dt\, ds = \tilde{R}_X(\omega_1, -\omega_2)$$

in view of Eq. (*6.93*).

**6.42.** Derive Eqs. (*6.98*) and (*6.99*).

Since $X(t)$ is WSS, by Eq. (*6.93*), and letting $t - s = \tau$, we have

$$\tilde{R}_X(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(t - s)e^{-j(\omega_1 t + \omega_2 s)}\, dt\, ds$$

$$= \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega_1 \tau}\, dt \int_{-\infty}^{\infty} e^{-j(\omega_1 + \omega_2)s}\, ds$$

$$= S_X(\omega_1)\int_{-\infty}^{\infty} e^{-j(\omega_1 + \omega_2)s}\, ds$$

From the Fourier transform pair (Appendix B) $1 \leftrightarrow 2\pi\delta(\omega)$, we have

$$\int_{-\infty}^{\infty} e^{-j\omega t}\, dt = 2\pi\delta(\omega)$$

Hence

$$\tilde{R}_X(\omega_1, \omega_2) = 2\pi S_X(\omega_1)\delta(\omega_1 + \omega_2)$$

Next, from Eq. (*6.94*) and the above result, we obtain

$$R_{\tilde{X}}(\omega_1, \omega_2) = \tilde{R}_X(\omega_1, -\omega_2) = 2\pi S_X(\omega_1)\delta(\omega_1 - \omega_2)$$

**6.43.** Let $\tilde{X}(\omega)$ be the Fourier transform of a random process $X(t)$. If $\tilde{X}(\omega)$ is a white noise with zero mean and autocorrelation function $q(\omega_1)\delta(\omega_1 - \omega_2)$, then show that $X(t)$ is WSS with power spectral density $q(\omega)/2\pi$.

By Eq. (*6.91*),

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}(\omega)e^{j\omega t}\, d\omega$$

Then
$$E[X(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} E[\tilde{X}(\omega)]e^{j\omega t}\, d\omega = 0 \tag{6.177}$$

Assuming that $X(t)$ is a complex random process, we have

$$\begin{aligned} R_X(t, s) = E[X(t)X^*(s)] &= E\left[\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{X}(\omega_1)\tilde{X}^*(\omega_2)e^{j(\omega_1 t - \omega_2 s)}\, d\omega_1\, d\omega_2\right] \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[\tilde{X}(\omega_1)\tilde{X}^*(\omega_2)]e^{j(\omega_1 t - \omega_2 s)}\, d\omega_1\, d\omega_2 \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q(\omega_1)\delta(\omega_1 - \omega_2)e^{j(\omega_1 t - \omega_2 s)}\, d\omega_1\, d\omega_2 \\ &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} q(\omega_1)e^{j\omega_1(t-s)}\, d\omega_1 \end{aligned} \tag{6.178}$$

which depends only on $t - s = \tau$. Hence, we conclude that $X(t)$ is WSS. Setting $t - s = \tau$ and $\omega_1 = \omega$ in Eq. (*6.178*), we have

$$\begin{aligned} R_X(\tau) &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty} q(\omega)e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2\pi}q(\omega)\right]e^{j\omega\tau}\, d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}\, d\omega \end{aligned}$$

in view of Eq. (*6.24*). Thus, we obtain $S_X(\omega) = q(\omega)/2\pi$.

**6.44.** Verify Eq. (*6.104*).

By Eq. (*6.100*),

$$\tilde{X}(\Omega_1) = \sum_{n=-\infty}^{\infty} X(n)e^{-j\Omega_1 n} \qquad \tilde{X}^*(\Omega_2) = \sum_{m=-\infty}^{\infty} X^*(m)e^{j\Omega_2 m}$$

Then
$$\begin{aligned} R_{\tilde{X}}(\Omega_1, \Omega_2) = E[\tilde{X}(\Omega_1)\tilde{X}^*(\Omega_2)] &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} E[X(n)X^*(m)]e^{-j(\Omega_1 n - \Omega_2 m)} \\ &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_X(n, m)e^{-j[\Omega_1 n + (-\Omega_2)m]} = \tilde{R}_X(\Omega_1, -\Omega_2) \end{aligned}$$

in view of Eq. (*6.103*).

**6.45.** Derive Eqs. (*6.105*) and (*6.106*).

If $X(n)$ is WSS, then $R_X(n, m) = R_X(n - m)$. By Eq. (*6.103*), and letting $n - m = k$, we have

$$\begin{aligned} \tilde{R}_X(\Omega_1, \Omega_2) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} R_X(n - m)e^{-j(\Omega_1 n + \Omega_2 m)} \\ &= \sum_{k=-\infty}^{\infty} R_X(k)e^{-j\Omega_1 k}\sum_{m=-\infty}^{\infty} e^{-j(\Omega_1 + \Omega_2)m} \\ &= S_X(\Omega_1)\sum_{m=-\infty}^{\infty} e^{-j(\Omega_1 + \Omega_2)m} \end{aligned}$$

From the Fourier transform pair (Appendix B) $x(n) = 1 \leftrightarrow 2\pi\delta(\Omega)$, we have

$$\sum_{m=-\infty}^{\infty} e^{-jm(\Omega_1+\Omega_2)} = 2\pi\delta(\Omega_1 + \Omega_2)$$

Hence
$$\tilde{R}_X(\Omega_1, \Omega_2) = 2\pi S_X(\Omega_1)\delta(\Omega_1 + \Omega_2)$$

Next, from Eq. (*6.104*) and the above result, we obtain

$$R_{\tilde{X}}(\Omega_1, \Omega_2) = \tilde{R}_X(\Omega_1, -\Omega_2) = 2\pi S_X(\Omega_1)\delta(\Omega_1 - \Omega_2)$$

## Supplementary Problems

**6.46.** Is the Poisson process $X(t)$ m.s. continuous?

*Hint*: Use Eq. (*5.60*) and proceed as in Prob. 6.4.

*Ans.* Yes.

**6.47.** Let $X(t)$ be defined by (Prob. 5.4)

$$X(t) = Y \cos \omega t \qquad t \geq 0$$

where $Y$ is a uniform r.v. over (0, 1) and $\omega$ is a constant.

(*a*) Is $X(t)$ m.s. continuous?

(*b*) Does $X(t)$ have a m.s. derivative?

*Hint*: Use Eq. (*5.87*) of Prob. 5.12.

*Ans.* (*a*) Yes; (*b*) yes.

**6.48.** Let $Z(t)$ be the random telegraph signal of Prob. 6.18.

(*a*) Is $Z(t)$ m.s. continuous?

(*b*) Does $Z(t)$ have a m.s. derivative?

*Hint*: Use Eq. (*6.132*) of Prob. 6.18.

*Ans.* (*a*) Yes; (*b*) no.

**6.49.** Let $X(t)$ be a WSS random process, and let $X'(t)$ be its m.s. derivative. Show that $E[X(t)X'(t)] = 0$.

*Hint*: Use Eqs. (*6.13*) [or (*6.14*)] and (*6.117*).

**6.50.** Let
$$Z(t) = \frac{2}{T}\int_{t}^{t+T/2} X(\alpha)\, d\alpha$$

where $X(t)$ is given by Prob. 6.47 with $\omega = 2\pi/T$.

(*a*) Find the mean of $Z(t)$.

(*b*) Find the autocorrelation function of $Z(t)$.

*Ans.* (*a*) $-\dfrac{1}{\pi}\sin \omega t$

(*b*) $R_Z(t, s) = \dfrac{4}{3\pi^2}\sin \omega t \sin \omega s$

**6.51.** Consider a WSS random process $X(t)$ with $E[X(t)] = \mu_X$. Let

$$\langle X(t)\rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} X(t)\,dt$$

The process $X(t)$ is said to be *ergodic in the mean* if

$$\text{l.i.m.}_{T\to\infty} \langle X(t)\rangle_T = E[X(t)] = \mu_X$$

Find $E[\langle X(t)\rangle_T]$.

*Ans.* $\mu_X$

**6.52.** Let $X(t) = A\cos(\omega_0 t + \Theta)$, where $A$ and $\omega_0$ are constants, $\Theta$ is a uniform r.v. over $(-\pi, \pi)$ (Prob. 5.20). Find the power spectral density of $X(t)$.

*Ans.* $S_X(\omega) = \dfrac{A^2\pi}{2}[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$

**6.53.** A random process $Y(t)$ is defined by

$$Y(t) = AX(t)\cos(\omega_c t + \Theta)$$

where $A$ and $\omega_c$ are constants, $\Theta$ is a uniform r.v. over $(-\pi, \pi)$, and $X(t)$ is a zero-mean WSS random process with the autocorrelation function $R_X(\tau)$ and the power spectral density $S_X(\omega)$. Furthermore, $X(t)$ and $\Theta$ are independent. Show that $Y(t)$ is WSS, and find the power spectral density of $Y(t)$.

*Ans.* $S_Y(\omega) = \dfrac{A^2}{4}[S_X(\omega - \omega_c) + S_X(\omega + \omega_c)]$

**6.54.** Consider a discrete-time random process defined by

$$X(n) = \sum_{i=1}^{m} a_i \cos(\Omega_i n + \Theta_i)$$

where $a_i$ and $\Omega_i$ are real constants and $\Theta_i$ are independent uniform r.v.'s over $(-\pi, \pi)$.

(*a*) Find the mean of $X(n)$.

(*b*) Find the autocorrelation function of $X(n)$.

*Ans.* (*a*) $E[X(n)] = 0$

(*b*) $R_X(n, n+k) = \dfrac{1}{2}\sum_{i=1}^{m} a_i^2 \cos(\Omega_i k)$

**6.55.** Consider a discrete-time WSS random process $X(n)$ with the autocorrelation function

$$R_X(k) = 10e^{-0.5|k|}$$

Find the power spectral density of $X(n)$.

*Ans.* $S_X(\Omega) = \dfrac{6.32}{1.368 - 1.213\cos\Omega} \qquad -\pi < \Omega < \pi$

**6.56.** Let $X(t)$ and $Y(t)$ be defined by

$$X(t) = U\cos\omega_0 t + V\sin\omega_0 t$$
$$Y(t) = V\cos\omega_0 t - U\sin\omega_0 t$$

where $\omega_0$ is constant and $U$ and $V$ are independent r.v.'s both having zero mean and variance $\sigma^2$.

(*a*) Find the cross-correlation function of $X(t)$ and $Y(t)$.

(*b*) Find the cross power spectral density of $X(t)$ and $Y(t)$.

*Ans.* (*a*) $R_{XY}(t, t+\tau) = -\sigma^2 \sin \omega_0 \tau$

(*b*) $S_{XY}(\omega) - j\sigma^2\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$

**6.57.** Verify Eqs. (*6.36*) and (*6.37*).

*Hint:* Substitute Eq. (*6.18*) into Eq. (*6.34*).

**6.58.** Let $Y(t) = X(t) + W(t)$, where $X(t)$ and $W(t)$ are orthogonal and $W(t)$ is a white noise specified by Eq. (*6.43*) or (*6.45*). Find the autocorrelation function of $Y(t)$.

*Ans.* $R_Y(t, s) = R_X(t, s) + \sigma^2\delta(t - s)$

**6.59.** A zero-mean WSS random process $X(t)$ is called band-limited white noise if its spectral density is given by

$$S_X(\omega) = \begin{cases} N_0/2 & |\omega| \le \omega_B \\ 0 & |\omega| > \omega_B \end{cases}$$

Find the autocorrelation function of $X(t)$.

*Ans.* $R_X(\tau) = \dfrac{N_0 \omega_B}{2\pi} \dfrac{\sin \omega_B \tau}{\omega_B \tau}$

**6.60.** A WSS random process $X(t)$ is applied to the input of an LTI system with impulse response $h(t) = 3e^{-2t}u(t)$. Find the mean value of $Y(t)$ of the system if $E[X(t)] = 2$.

*Hint:* Use Eq. (*6.59*).

*Ans.* 3

**6.61.** The input $X(t)$ to the $RC$ filter shown in Fig. 6-7 is a white noise specified by Eq. (*6.45*). Find the mean-square value of $Y(t)$.

*Hint:* Use Eqs. (*6.64*) and (*6.65*).

*Ans.* $\sigma^2/(2RC)$

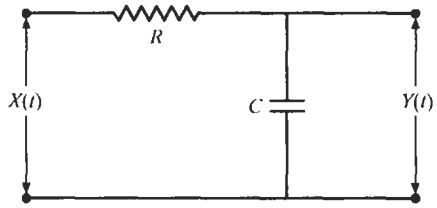



**Fig. 6-7** *RC* filter.

**6.62.** The input $X(t)$ to a differentiator is the random telegraph signal of Prob. 6.18.

(*a*) Determine the power spectral density of the differentiator output.

(*b*) Find the mean-square value of the differentiator output.

*Ans.* (*a*) $S_Y(\omega) = \dfrac{4\lambda\omega^2}{\omega^2 + 4\lambda^2}$

(*b*) $E[Y^2(t)] = \infty$

**6.63.** Suppose that the input to the filter shown in Fig. 6-8 is a white noise specified by Eq. (*6.45*). Find the power spectral density of $Y(t)$.

*Ans.* $S_Y(\omega) = \sigma^2(1 + a^2 + 2a \cos \omega T)$

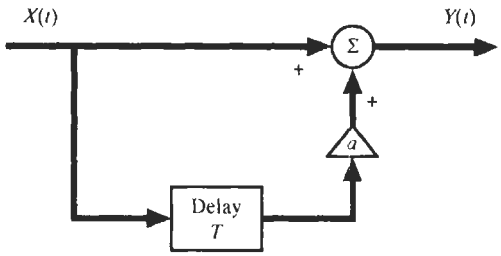



Fig. 6-8

**6.64.** Verify Eq. (6.67).

*Hint:* Proceed as in Prob. 6.24.

**6.65.** Suppose that the input to the discrete-time filter shown in Fig. 6-9 is a discrete-time white noise with average power $\sigma^2$. Find the power spectral density of $Y(n)$.

*Ans.* $S_Y(\Omega) = \sigma^2(1 + a^2 + 2a \cos \Omega)$

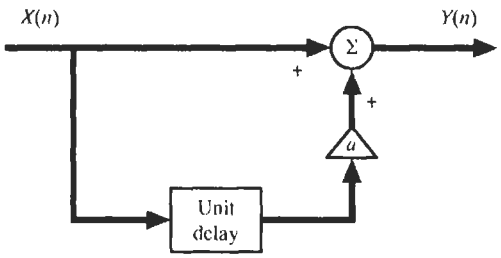



Fig. 6-9

**6.66.** Using the Karhunen-Loève expansion of the Wiener process, obtain the Karhunen-Loève expansion of the white normal noise.

*Hint:* Take the derivative of Eq. (6.175) of Prob. 6.39.

*Ans.* $\sqrt{\frac{2}{T}} \sum_{n=1}^{\infty} W_n \cos\left(n - \frac{1}{2}\right) \frac{\pi}{T} t \qquad 0 < t < T$

where $W_n$ are independent normal r.v.'s with the same variance $\sigma^2$.

**6.67.** Let $Y(t) = X(t) + W(t)$, where $X(t)$ and $W(t)$ are orthogonal and $W(t)$ is a white noise specified by Eq. (6.43) or (6.45). Let $\phi_n(t)$ be the eigenfunctions of the integral equation (6.86) and $\lambda_n$ the corresponding eigenvalues.

(*a*) Show that $\phi_n(t)$ are also the eigenfunctions of the integral equation for the Karhunen-Loève expansion of $Y(t)$ with $R_Y(t, s)$.

(*b*) Find the corresponding eigenvalues.

*Hint:* Use the result of Prob. 6.58.

*Ans.* (*b*) $\lambda_n + \sigma^2$

**6.68.** Suppose that

$$X(t) = \sum_n X_n e^{jn\omega_0 t}$$

where $X_n$ are r.v.'s and $\omega_0$ is a constant. Find the Fourier transform of $X(t)$.

*Ans.* $\tilde{X}(\omega) = \sum_n 2\pi X_n \delta(\omega - n\omega_0)$

**6.69.** Let $\tilde{X}(\omega)$ be the Fourier transform of a continuous-time random process $X(t)$. Find the mean of $\tilde{X}(\omega)$.

*Ans.* $\mathcal{F}[\mu_X(t)] = \int_{-\infty}^{\infty} \mu_X(t) e^{-j\omega t}\, dt$ where $\mu_X(t) = E[X(t)]$

**6.70.** Let

$$\tilde{X}(\Omega) = \sum_{n=-\infty}^{\infty} X(n) e^{-j\Omega n}$$

where $E[X(n)] = 0$ and $E[X(n)X(k)] = \sigma_n^2\, \delta(n-k)$. Find the mean and the autocorrelation function of $\tilde{X}(\Omega)$.

*Ans.* $E[\tilde{X}(\Omega)] = 0$ $\qquad R_{\tilde{X}}(\Omega_1, \Omega_2) = \sum_{n=-\infty}^{\infty} \sigma_n^2 e^{-j(\Omega_1 - \Omega_2)n}$

# Chapter 7

# Estimation Theory

## 7.1 INTRODUCTION

In this chapter, we present a classical estimation theory. There are two basic types of estimation problems. In the first type, we are interested in estimating the parameters of one or more r.v.'s, and in the second type, we are interested in estimating the value of an inaccessible r.v. $Y$ in terms of the observation of an accessible r.v. $X$.

## 7.2 PARAMETER ESTIMATION

Let $X$ be a r.v. with pdf $f(x)$ and $X_1, \ldots, X_n$ a set of $n$ independent r.v.'s each with pdf $f(x)$. The set of r.v.'s $(X_1, \ldots, X_n)$ is called a *random sample* (or *sample vector*) of size $n$ of $X$. Any real-valued function of a random sample $s(X_1, \ldots, X_n)$ is called a *statistic*.

Let $X$ be a r.v. with pdf $f(x; \theta)$ which depends on an unknown parameter $\theta$. Let $(X_1, \ldots, X_n)$ be a random sample of $X$. In this case, the joint pdf of $X_1, \ldots, X_n$ is given by

$$f(\mathbf{x}; \theta) = f(x_1, \ldots, x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta) \tag{7.1}$$

where $x_1, \ldots, x_n$ are the values of the observed data taken from the random sample.

An *estimator* of $\theta$ is any statistic $s(X_1, \ldots, X_n)$, denoted as

$$\Theta = s(X_1, \ldots, X_n) \tag{7.2}$$

For a particular set of observations $X_1 = x_1, \ldots, X_n = x_n$, the value of the estimator $s(x_1, \ldots, x_n)$ will be called an *estimate* of $\theta$ and denoted by $\hat{\theta}$. Thus an estimator is a r.v. and an estimate is a particular realization of it. It is not necessary that an estimate of a parameter be one single value; instead, the estimate could be a *range* of values. Estimates which specify a single value are called *point estimates*, and estimates which specify a range of values are called *interval estimates*.

## 7.3 PROPERTIES OF POINT ESTIMATORS

### A. Unbiased Estimators:

An estimator $\Theta = s(X_1, \ldots, X_n)$ is said to be an *unbiased* estimator of the parameter $\theta$ if

$$E(\Theta) = \theta \tag{7.3}$$

for all possible values of $\theta$. If $\Theta$ is an unbiased estimator, then its mean square error is given by

$$E[(\Theta - \theta)^2] = E\{[\Theta - E(\Theta)]^2\} = \mathrm{Var}(\Theta) \tag{7.4}$$

That is, its mean square error equals its variance.

### B. Efficient Estimators:

An estimator $\Theta_1$ is said to be a more *efficient* estimator of the parameter $\theta$ than the estimator $\Theta_2$ if

1. $\Theta_1$ and $\Theta_2$ are both unbiased estimators of $\theta$.

2. $\mathrm{Var}(\Theta_1) < \mathrm{Var}(\Theta_2)$.

The estimator $\Theta_{MV} = s(X_1, \ldots, X_n)$ is said to be a *most efficient* (or *minimum variance*) unbiased estimator of the parameter $\theta$ if

1. It is an unbiased estimator of $\theta$.
2. $\mathrm{Var}(\Theta_{MV}) \leq \mathrm{Var}(\Theta)$ for all $\Theta$.

### C. Consistent Estimators:

The estimator $\Theta_n$ of $\theta$ based on a random sample of size $n$ is said to be *consistent* if for any small $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|\Theta_n - \theta| < \varepsilon) = 1 \tag{7.5}$$

or equivalently,

$$\lim_{n\to\infty} P(|\Theta_n - \theta| \geq \varepsilon) = 0 \tag{7.6}$$

The following two conditions are sufficient to define consistency (Prob. 7.5):

1. $\lim_{n\to\infty} E(\Theta_n) = \theta$ (7.7)
2. $\lim_{n\to\infty} \mathrm{Var}(\Theta_n) = 0$ (7.8)

## 7.4 MAXIMUM-LIKELIHOOD ESTIMATION

Let $f(\mathbf{x}; \theta) = f(x_1, \ldots, x_n; \theta)$ denote the joint pmf of the r.v.'s $X_1, \ldots, X_n$ when they are discrete, and let it be their joint pdf when they are continuous. Let

$$L(\theta) = f(\mathbf{x}; \theta) = f(x_1, \ldots, x_n; \theta) \tag{7.9}$$

Now $L(\theta)$ represents the likelihood that the values $x_1, \ldots, x_n$ will be observed when $\theta$ is the true value of the parameter. Thus $L(\theta)$ is often referred to as the *likelihood function* of the random sample. Let $\theta_{ML} = s(x_1, \ldots, x_n)$ be the maximizing value of $L(\theta)$; that is,

$$L(\theta_{ML}) = \max_{\theta} L(\theta) \tag{7.10}$$

Then the maximum-likelihood estimator of $\theta$ is

$$\Theta_{ML} = s(X_1, \ldots, X_n) \tag{7.11}$$

and $\theta_{ML}$ is the maximum-likelihood estimate of $\theta$.

Since $L(\theta)$ is a product of either pmf's or pdf's, it will always be positive (for the range of possible value of $\theta$). Thus $\ln L(\theta)$ can always be defined, and in determining the maximizing value of $\theta$, it is often useful to use the fact that $L(\theta)$ and $\ln L(\theta)$ have their maximum at the same value of $\theta$. Hence, we may also obtain $\theta_{ML}$ by maximizing $\ln L(\theta)$.

## 7.5 BAYES' ESTIMATION

Suppose that the unknown parameter $\theta$ is considered to be a r.v. having some fixed distribution or *prior* pdf $f(\theta)$. Then $f(x; \theta)$ is now viewed as a conditional pdf and written as $f(x \mid \theta)$, and we can express the joint pdf of the random sample $(X_1, \ldots, X_n)$ and $\theta$ as

$$f(x_1, \ldots, x_n, \theta) = f(x_1, \ldots, x_n \mid \theta) f(\theta) \tag{7.12}$$

and the marginal pdf of the sample is given by

$$f(x_1, \ldots, x_n) = \int_{R_\theta} f(x_1, \ldots, x_n, \theta)\, d\theta \tag{7.13}$$

where $R_\theta$ is the range of the possible value of $\theta$. The other conditional pdf,

$$f(\theta \mid x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n, \theta)}{f(x_1, \ldots, x_n)} = \frac{f(x_1, \ldots, x_n \mid \theta) f(\theta)}{f(x_1, \ldots, x_n)} \tag{7.14}$$

is referred to as the *posterior* pdf of $\theta$. Thus the prior pdf $f(\theta)$ represents our information about $\theta$ prior to the observation of the outcomes of $X_1, \ldots, X_n$, and the posterior pdf $f(\theta \mid x_1, \ldots, x_n)$ represents our information about $\theta$ after having observed the sample.

The conditional mean of $\theta$, defined by

$$\theta_B = E(\theta \mid x_1, \ldots, x_n) = \int_{R_\theta} \theta f(\theta \mid x_1, \ldots, x_n)\, d\theta \tag{7.15}$$

is called the *Bayes' estimate* of $\theta$, and

$$\Theta_B = E(\theta \mid X_1, \ldots, X_n) \tag{7.16}$$

is called the *Bayes' estimator* of $\theta$.

## 7.6 MEAN SQUARE ESTIMATION

In this section, we deal with the second type of estimation problem—that is, estimating the value of an inaccessible r.v. $Y$ in terms of the observation of an accessible r.v. $X$. In general, the estimator $\hat{Y}$ of $Y$ is given by a function of $X$, $g(X)$. Then $Y - \hat{Y} = Y - g(X)$ is called the *estimation error*, and there is a *cost* associated with this error, $C[Y - g(X)]$. We are interested in finding the function $g(X)$ that minimizes this cost. When $X$ and $Y$ are continuous r.v.'s, the *mean square* (m.s.) *error* is often used as the cost function,

$$C[Y - g(X)] = E\{[Y - g(X)]^2\} \tag{7.17}$$

It can be shown that the estimator of $Y$ given by (Prob. 7.17),

$$\hat{Y} = g(X) = E(Y \mid X) \tag{7.18}$$

is the best estimator in the sense that the m.s. error defined by Eq. (7.17) is a minimum.

## 7.7 LINEAR MEAN SQUARE ESTIMATION

Now consider the estimator $\hat{Y}$ of $Y$ given by

$$\hat{Y} = g(X) = aX + b \tag{7.19}$$

We would like to find the values of $a$ and $b$ such that the m.s. error defined by

$$e = E[(Y - \hat{Y})^2] = E\{[Y - (aX + b)]^2\} \tag{7.20}$$

is minimum. We maintain that $a$ and $b$ must be such that (Prob. 7.20)

$$E\{[Y - (aX + b)]X\} = 0 \tag{7.21}$$

and $a$ and $b$ are given by

$$a = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{\sigma_Y}{\sigma_X} \rho_{XY} \qquad b = \mu_Y - a\mu_X \tag{7.22}$$

and the minimum m.s. error $e_m$ is (Prob. 7.22)

$$e_m = \sigma_Y^2(1 - \rho_{XY}^2) \tag{7.23}$$

where $\sigma_{XY} = \text{Cov}(X, Y)$ and $\rho_{XY}$ is the correlation coefficient of $X$ and $Y$. Note that Eq. (7.21) states that the optimum linear m.s. estimator $\hat{Y} = aX + b$ of $Y$ is such that the estimation error $Y - \hat{Y} = Y - (aX + b)$ is orthogonal to the observation $X$. This is known as the *orthogonality principle*. The line $y = ax + b$ is often called a *regression line*.

Next, we consider the estimator $\hat{Y}$ of $Y$ with a linear combination of the random sample $(X_1, \ldots, X_n)$ by

$$\hat{Y} = \sum_{i=1}^{n} a_i X_i \tag{7.24}$$

Again, we maintain that in order to produce the linear estimator with the minimum m.s. error, the coefficients $a_i$ must be such that the following orthogonality conditions are satisfied (Prob. 7.35):

$$E\left[\left(Y - \sum_{i=1}^{n} a_i X_i\right)X_j\right] = 0 \qquad j = 1, \ldots, n \tag{7.25}$$

Solving Eq. (7.25) for $a_i$, we obtain

$$\mathbf{a} = R^{-1}\mathbf{b} \tag{7.26}$$

where

$$\mathbf{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \qquad b_j = E(YX_j) \qquad R = \begin{bmatrix} R_{11} & \cdots & R_{1n} \\ \vdots & \ddots & \vdots \\ R_{n1} & \cdots & R_{nn} \end{bmatrix} \qquad R_{ij} = E(X_i X_j)$$

and $R^{-1}$ is the inverse of $R$.

## Solved Problems

### PROPERTIES OF POINT ESTIMATORS

**7.1.** Let $(X_1, \ldots, X_n)$ be a random sample of $X$ having unknown mean $\mu$. Show that the estimator of $\mu$ defined by

$$M = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \tag{7.27}$$

is an unbiased estimator of $\mu$. Note that $\bar{X}$ is known as the *sample mean* (Prob. 4.64).

By Eq. (4.108),

$$E(M) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}(n\mu) = \mu$$

Thus, $M$ is an unbiased estimator of $\mu$.

**7.2.** Let $(X_1, \ldots, X_n)$ be a random sample of $X$ having unknown mean $\mu$ and variance $\sigma^2$. Show that the estimator of $\sigma^2$ defined by

$$S^2 = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{7.28}$$

where $\bar{X}$ is the sample mean, is a biased estimator of $\sigma^2$.

By definition, we have

$$\sigma^2 = E[(X_i - \mu)^2]$$

Now
$$E(S^2) = E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = E\left\{\frac{1}{n}\sum_{i=1}^{n}[(X_i - \mu) - (\bar{X} - \mu)]^2\right\}$$
$$= E\left\{\frac{1}{n}\sum_{i=1}^{n}[(X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) + (\bar{X} - \mu)^2]\right\}$$
$$= E\left\{\frac{1}{n}\left[\sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2\right]\right\}$$
$$= \frac{1}{n}\sum_{i=1}^{n}E[(X_i - \mu)^2] - E[(\bar{X} - \mu)^2] = \sigma^2 - \sigma_{\bar{X}}^2$$

By Eqs. (*4.112*) and (*7.27*), we have

$$\sigma_{\bar{X}}^2 = \mathrm{Var}(\bar{X}) = \sum_{i=1}^{n}\frac{1}{n^2}\sigma^2 = \frac{1}{n}\sigma^2$$

Thus
$$E(S^2) = \sigma^2 - \frac{1}{n}\sigma^2 = \frac{n-1}{n}\sigma^2 \tag{7.29}$$

which shows that $S^2$ is a biased estimator of $\sigma^2$.

**7.3.** Let $(X_1, \ldots, X_n)$ be a random sample of a Poisson r.v. $X$ with unknown parameter $\lambda$.

(*a*) Show that

$$\Lambda_1 = \frac{1}{n}\sum_{i=1}^{n}X_i \qquad \text{and} \qquad \Lambda_2 = \tfrac{1}{2}(X_1 + X_2)$$

are both unbiased estimators of $\lambda$.

(*b*) Which estimator is more efficient?

(*a*) By Eqs. (*2.42*) and (*4.108*), we have

$$E(\Lambda_1) = \frac{1}{n}\sum_{i=1}^{n}E(X_i) = \frac{1}{n}(n\lambda) = \lambda$$
$$E(\Lambda_2) = \tfrac{1}{2}[E(X_1) + E(X_2)] = \tfrac{1}{2}(2\lambda) = \lambda$$

Thus, both estimators are unbiased estimators of $\lambda$.

(*b*) By Eqs. (*2.43*) and (*4.112*),

$$\mathrm{Var}(\Lambda_1) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i) = \frac{1}{n^2}(n\lambda) = \frac{\lambda}{n}$$
$$\mathrm{Var}(\Lambda_2) = \tfrac{1}{4}(2\lambda) = \frac{\lambda}{2}$$

Thus, if $n > 2$, $\Lambda_1$ is a more efficient estimator of $\lambda$ than $\Lambda_2$, since $\lambda/n < \lambda/2$.

**7.4.** Let $(X_1, \ldots, X_n)$ be a random sample of $X$ with mean $\mu$ and variance $\sigma^2$. A linear estimator of $\mu$ is defined to be a linear function of $X_1, \ldots, X_n$, $l(X_1, \ldots, X_n)$. Show that the linear estimator defined by [Eq. (*7.27*)],

$$M = \frac{1}{n}\sum_{i=1}^{n}X_i = \bar{X}$$

is the most efficient linear unbiased estimator of $\mu$.

Assume that

$$M_1 = l(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i$$

is a linear unbiased estimator of $\mu$ with lower variance than $M$. Since $M_1$ is unbiased, we must have

$$E(M_1) = \sum_{i=1}^{n} a_i E(X_i) = \mu \sum_{i=1}^{n} a_i = \mu$$

which implies that $\sum_{i=1}^{n} a_i = 1$. By Eq. (*4.112*),

$$\mathrm{Var}(M) = \frac{1}{n}\sigma^2 \qquad \text{and} \qquad \mathrm{Var}(M_1) = \sigma^2 \sum_{i=1}^{n} a_i^2$$

By assumption,

$$\sigma^2 \sum_{i=1}^{n} a_i^2 < \frac{1}{n}\sigma^2 \qquad \text{or} \qquad \sum_{i=1}^{n} a_i^2 < \frac{1}{n} \tag{7.30}$$

Consider the sum

$$\begin{aligned} 0 \le \sum_{i=1}^{n} \left(a_i - \frac{1}{n}\right)^2 &= \sum_{i=1}^{n} \left(a_i^2 - 2\frac{a_i}{n} + \frac{1}{n^2}\right) \\ &= \sum_{i=1}^{n} a_i^2 - \frac{2}{n}\sum_{i=1}^{n} a_i + \frac{1}{n} \\ &= \sum_{i=1}^{n} a_i^2 - \frac{1}{n} \end{aligned}$$

which, by assumption (*7.30*), is less than 0. This is impossible unless $a_i = 1/n$, implying that $M$ is the most efficient linear unbiased estimator of $\mu$.

**7.5.** Show that if

$$\lim_{n\to\infty} E(\Theta_n) = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{Var}(\Theta_n) = 0$$

then the estimator $\Theta_n$ is consistent.

Using Chebyshev's inequality (*2.97*), we can write

$$\begin{aligned} P(|\Theta_n - \theta| \ge \varepsilon) &\le \frac{E[(\Theta_n - \theta)^2]}{\varepsilon^2} = \frac{1}{\varepsilon^2} E\{[\Theta_n - E(\Theta_n) + E(\Theta_n) - \theta]^2\} \\ &= \frac{1}{\varepsilon^2} E\{[\Theta_n - E(\Theta_n)]^2 + [E(\Theta_n) - \theta]^2 + 2[\Theta_n - E(\Theta_n)][E(\Theta_n) - \theta]\} \\ &= \frac{1}{\varepsilon^2} (\mathrm{Var}(\Theta_n) + E\{[E(\Theta_n) - \theta]^2\} + 2E\{[\Theta_n - E(\Theta_n)][E(\Theta_n) - \theta]\}) \end{aligned}$$

Thus, if

$$\lim_{n\to\infty} E(\Theta_n) = \theta \qquad \text{and} \qquad \lim_{n\to\infty} \mathrm{Var}(\Theta_n) = 0$$

then

$$\lim_{n\to\infty} P(|\Theta_n - \theta| \ge \varepsilon) = 0$$

that is, $\Theta_n$ is consistent [see Eq. (*7.6*)].

**7.6.** Let $(X_1, \ldots, X_n)$ be a random sample of a uniform r.v. $X$ over $(0, a)$, where $a$ is unknown. Show that

$$A = \max(X_1, X_2, \ldots, X_n) \tag{7.31}$$

is a consistent estimator of the parameter $a$.

If $X$ is uniformly distributed over $(0, a)$, then from Eqs. (*2.44*), (*2.45*), and (*4.98*) of Prob. 4.30, the pdf of $Z = \max(X_1, \ldots, X_n)$ is

$$f_Z(z) = nf_X(z)[F_X(z)]^{n-1} = \frac{n}{a}\left(\frac{z}{a}\right)^{n-1} \qquad 0 < z < a \tag{7.32}$$

Thus
$$E(A) = \int_0^a z f_Z(z)\, dz = \frac{n}{a^n}\int_0^a z^n\, dz = \frac{n}{n+1}\, a$$

and
$$\lim_{n\to\infty} E(A) = a$$

Next,
$$E(A^2) = \int_0^a z^2 f_Z(z)\, dz = \frac{n}{a^n}\int_0^a z^{n+1}\, dz = \frac{n}{n+2}\, a^2$$

$$\mathrm{Var}(A) = E(A^2) - [E(A)]^2 = \frac{na^2}{n+2} - \frac{n^2a^2}{(n+1)^2} = \frac{n}{(n+2)(n+1)^2}\, a^2$$

and
$$\lim_{n\to\infty} \mathrm{Var}(A) = 0$$

Thus, by Eqs. (*7.7*) and (*7.8*), $A$ is a consistent estimator of parameter $a$.

## MAXIMUM-LIKELIHOOD ESTIMATION

**7.7.** Let $(X_1, \ldots, X_n)$ be a random sample of a binomial r.v. $X$ with parameters $(m, p)$, where $m$ is assumed to be known and $p$ unknown. Determine the maximum-likelihood estimator of $p$.

The likelihood function is given by [Eq. (*2.36*)]

$$L(p) = f(x_1, \ldots, x_n; p) = \binom{m}{x_1} p^{x_1}(1-p)^{(m-x_1)} \cdots \binom{m}{x_n} p^{x_n}(1-p)^{(m-x_n)}$$

$$= \binom{m}{x_1} \cdots \binom{m}{x_n} p^{\sum_{i=1}^n x_i}(1-p)^{(mn - \sum_{i=1}^n x_i)}$$

Taking the natural logarithm of the above expression, we get

$$\ln L(p) = \ln c + \left(\sum_{i=1}^n x_i\right) \ln p + \left(mn - \sum_{i=1}^n x_i\right) \ln(1-p)$$

where
$$c = \prod_{i=1}^n \binom{m}{x_i}$$

and
$$\frac{d}{dp} \ln L(p) = \frac{1}{p}\sum_{i=1}^n x_i - \frac{1}{1-p}\left(mn - \sum_{i=1}^n x_i\right)$$

Setting $d[\ln L(p)]/dp = 0$, the maximum-likelihood estimate $\hat{p}_{ML}$ of $p$ is obtained as

$$\frac{1}{\hat{p}_{ML}} \sum_{i=1}^n x_i = \frac{1}{1-\hat{p}_{ML}}\left(mn - \sum_{i=1}^n x_i\right)$$

or
$$\hat{p}_{ML} = \frac{1}{mn}\sum_{i=1}^n x_i \tag{7.33}$$

Hence, the maximum-likelihood estimator of $p$ is given by

$$P_{ML} = \frac{1}{mn}\sum_{i=1}^n X_i = \frac{1}{m}\bar{X} \tag{7.34}$$

**7.8.** Let $(X_1, \ldots, X_n)$ be a random sample of a Poisson r.v. with unknown parameter $\lambda$. Determine the maximum-likelihood estimator of $\lambda$.

The likelihood function is given by [Eq. (2.40)]

$$L(\lambda) = f(x_1, \ldots, x_n; \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{x_1! \cdots x_n!}$$

Thus,
$$\ln L(\lambda) = -n\lambda + \ln \lambda \sum_{i=1}^{n} x_i - \ln c$$

where
$$c = \prod_{i=1}^{n} (x_i!)$$

and
$$\frac{d}{d\lambda} \ln L(\lambda) = -n + \frac{1}{\lambda} \sum_{i=1}^{n} x_i$$

Setting $d[\ln L(\lambda)]/d\lambda = 0$, the maximum-likelihood estimate $\hat{\lambda}_{ML}$ of $\lambda$ is obtained as

$$\hat{\lambda}_{ML} = \frac{1}{n} \sum_{i=1}^{n} x_i \tag{7.35}$$

Hence, the maximum-likelihood estimator of $\lambda$ is given by

$$\Lambda_{ML} = s(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X} \tag{7.36}$$

**7.9.** Let $(X_1, \ldots, X_n)$ be a random sample of an exponential r.v. $X$ with unknown parameter $\lambda$. Determine the maximum-likelihood estimator of $\lambda$.

The likelihood function is given by [Eq. (2.48)]

$$L(\lambda) = f(x_1, \ldots, x_n; \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i}$$

Thus,
$$\ln L(\lambda) = n \ln \lambda - \lambda \sum_{i=1}^{n} x_i$$

and
$$\frac{d}{d\lambda} \ln L(\lambda) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

Setting $d[\ln L(\lambda)]/d\lambda = 0$, the maximum-likelihood estimate $\hat{\lambda}_{ML}$ of $\lambda$ is obtained as

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i} \tag{7.37}$$

Hence, the maximum-likelihood estimator of $\lambda$ is given by

$$\Lambda_{ML} = s(X_1, \ldots, X_n) = \frac{n}{\sum_{i=1}^{n} X_i} = \frac{1}{\bar{X}} \tag{7.38}$$

**7.10.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal random r.v. $X$ with unknown mean $\mu$ and unknown variance $\sigma^2$. Determine the maximum-likelihood estimators of $\mu$ and $\sigma^2$.

The likelihood function is given by [Eq. (2.52)]

$$L(\mu, \sigma) = f(x_1, \ldots, x_n; \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{1}{2\sigma^2}(x_i - \mu)^2\right]$$

$$= \left(\frac{1}{2\pi}\right)^{n/2} \frac{1}{\sigma^n} \exp\left[-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \mu)^2\right]$$

Thus, $$\ln L(\mu, \sigma) = -\frac{n}{2}\ln(2\pi) - n \ln \sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

In order to find the values of $\mu$ and $\sigma$ maximizing the above, we compute

$$\frac{\partial}{\partial \mu} \ln L(\mu, \sigma) = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)$$

$$\frac{\partial}{\partial \sigma} \ln L(\mu, \sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2$$

Equating these equations to zero, we get

$$\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML}) = 0$$

and $$\frac{1}{\hat{\sigma}_{ML}^{\,3}}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML})^2 = \frac{n}{\hat{\sigma}_{ML}}$$

Solving for $\hat{\mu}_{ML}$ and $\hat{\sigma}_{ML}$, the maximum-likelihood estimates of $\mu$ and $\sigma^2$ are given, respectively, by

$$\hat{\mu}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{7.39}$$

$$\hat{\sigma}_{ML}^{\,2} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{\mu}_{ML})^2 \tag{7.40}$$

Hence, the maximum-likelihood estimators of $\mu$ and $\sigma^2$ are given, respectively, by

$$M_{ML} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X} \tag{7.41}$$

$$S_{ML}^{\,2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \tag{7.42}$$

## BAYES' ESTIMATION

**7.11.** Let $(X_1, \ldots, X_n)$ be the random sample of a Bernoulli r.v. $X$ with pmf given by [Eq. (2.32)]

$$f(x; p) = p^x(1 - p)^{1-x} \qquad x = 0, 1 \tag{7.43}$$

where $p$, $0 \le p \le 1$, is unknown. Assume that $p$ is a uniform r.v. over (0, 1). Find the Bayes' estimator of $p$.

The prior pdf of $p$ is the uniform pdf; that is,

$$f(p) = 1 \qquad 0 < p < 1$$

The posterior pdf of $p$ is given by

$$f(p \mid x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n, p)}{f(x_1, \ldots, x_n)}$$

Then, by Eq. (7.12),

$$\begin{aligned} f(x_1, \ldots, x_n, p) &= f(x_1, \ldots, x_n \mid p) f(p) \\ &= p^{\sum_{i=1}^{n} x_i}(1 - p)^{n - \sum_{i=1}^{n} x_i} = p^m(1 - p)^{n-m} \end{aligned}$$

where $m = \sum_{i=1}^{n} x_i$, and by Eq. (7.13),

$$f(x_1, \ldots, x_n) = \int_0^1 f(x_1, \ldots, x_n, p)\, dp = \int_0^1 p^m(1 - p)^{n-m}\, dp$$

Now, from calculus, for integers $m$ and $k$, we have

$$\int_0^1 p^m(1 - p)^k\, dp = \frac{m!\,k!}{(m + k + 1)!} \tag{7.44}$$

Thus, by Eq. (7.14), the posterior pdf of $p$ is

$$f(p \mid x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n, p)}{f(x_1, \ldots, x_n)} = \frac{(n+1)!\, p^m (1-p)^{n-m}}{m!(n-m)!}$$

and by Eqs. (7.15) and (7.44),

$$\begin{aligned} E(p \mid x_1, \ldots, x_n) &= \int_0^1 p f(p \mid x_1, \ldots, x_n)\, dp \\ &= \frac{(n+1)!}{m!(n-m)!} \int_0^1 p^{m+1}(1-p)^{n-m}\, dp \\ &= \frac{(n+1)!}{m!(n-m)!} \frac{(m+1)!(n-m)!}{(n+2)!} \\ &= \frac{m+1}{n+2} = \frac{1}{n+2}\left(\sum_{i=1}^{n} x_i + 1\right) \end{aligned}$$

Hence, by Eq. (7.16), the Bayes' estimator of $p$ is

$$P_B = E(p \mid X_1, \ldots, X_n) = \frac{1}{n+2}\left(\sum_{i=1}^{n} X_i + 1\right) \tag{7.45}$$

**7.12.** Let $(X_1, \ldots, X_n)$ be a random sample of an exponential r.v. $X$ with unknown parameter $\lambda$. Assume that $\lambda$ is itself to be an exponential r.v. with parameter $\alpha$. Find the Bayes' estimator of $\lambda$.

The assumed prior pdf of $\lambda$ is [Eq. (2.48)]

$$f(\lambda) = \begin{cases} \alpha e^{-\alpha\lambda} & \alpha, \lambda > 0 \\ 0 & \text{otherwise} \end{cases}$$

Now $$f(x_1, \ldots, x_n \mid \lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} x_i} = \lambda^n e^{-m\lambda}$$

where $m = \sum_{i=1}^{n} x_i$. Then, by Eqs. (7.12) and (7.13),

$$\begin{aligned} f(x_1, \ldots, x_n) &= \int_0^\infty f(x_1, \ldots, x_n \mid \lambda) f(\lambda)\, d\lambda \\ &= \int_0^\infty \lambda^n e^{-m\lambda} \alpha e^{-\alpha\lambda}\, d\lambda \\ &= \alpha \int_0^\infty \lambda^n e^{-(\alpha+m)\lambda}\, d\lambda = \alpha \frac{n!}{(\alpha+m)^{n+1}} \end{aligned}$$

By Eq. (7.14), the posterior pdf of $\lambda$ is given by

$$f(\lambda \mid x_1, \ldots, x_n) = \frac{f(x_1, \ldots, x_n \mid \lambda) f(\lambda)}{f(x_1, \ldots, x_n)} = \frac{(\alpha+m)^{n+1} \lambda^n e^{-(\alpha+m)\lambda}}{n!}$$

Thus, by Eq. (7.15), the Bayes' estimate of $\lambda$ is

$$\begin{aligned} \hat{\lambda}_B = E(\lambda \mid x_1, \ldots, x_n) &= \int_0^\infty \lambda f(\lambda \mid x_1, \ldots, x_n)\, d\lambda \\ &= \frac{(\alpha+m)^{n+1}}{n!} \int_0^\infty \lambda^{n+1} e^{-(\alpha+m)\lambda}\, d\lambda \\ &= \frac{(\alpha+m)^{n+1}}{n!} \frac{(n+1)!}{(\alpha+m)^{n+2}} \\ &= \frac{n+1}{\alpha+m} = \frac{n+1}{\alpha + \sum_{i=1}^{n} x_i} \end{aligned} \tag{7.46}$$

and the Bayes' estimator of $\lambda$ is

$$\Lambda_B = \frac{n+1}{\alpha + \sum_{i=1}^{n} X_i} = \frac{n+1}{\alpha + n\bar{X}} \tag{7.47}$$

**7.13.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. $X$ with unknown mean $\mu$ and variance 1. Assume that $\mu$ is itself to be a normal r.v. with mean 0 and variance 1. Find the Bayes' estimator of $\mu$.

The assumed prior pdf of $\mu$ is

$$f(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2}$$

Then by Eq. (7.12),

$$\begin{aligned} f(x_1, \ldots, x_n, \mu) &= f(x_1, \ldots, x_n \mid \mu) f(\mu) \\ &= \frac{1}{(2\pi)^{n/2}} \exp\left[ -\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{2} \right] \frac{1}{\sqrt{2\pi}} e^{-\mu^2/2} \\ &= \frac{\exp\left( -\sum_{i=1}^{n} \frac{x_i^2}{2} \right)}{(2\pi)^{(n+1)/2}} \exp\left[ -\frac{(n+1)}{2} \left( \mu^2 - \frac{2\mu}{n+1} \sum_{i=1}^{n} x_i \right) \right] \\ &= \frac{\exp\left( -\sum_{i=1}^{n} \frac{x_i^2}{2} \right) \exp\left[ \frac{1}{2(n+1)} \left( \sum_{i=1}^{n} x_i \right)^2 \right]}{(2\pi)^{(n+1)/2}} \exp\left[ -\frac{(n+1)}{2} \left( \mu - \frac{1}{n+1} \sum_{i=1}^{n} x_i \right)^2 \right] \end{aligned}$$

Then, by Eq. (7.14), the posterior pdf of $\mu$ is given by

$$\begin{aligned} f(\mu \mid x_1, \ldots, x_n) &= \frac{f(x_1, \ldots, x_n, \mu)}{\int_{-\infty}^{\infty} f(x_1, \ldots, x_n, \mu)\, d\mu} \\ &= C \exp\left[ -\frac{(n+1)}{2} \left( \mu - \frac{1}{n+1} \sum_{i=1}^{n} x_i \right)^2 \right] \end{aligned} \tag{7.48}$$

where $C = C(x_1, \ldots, x_n)$ is independent of $\mu$. However, Eq. (7.48) is just the pdf of a normal r.v. with mean

$$\frac{1}{n+1} \left( \sum_{i=1}^{n} x_i \right)$$

and variance

$$\frac{1}{n+1}$$

Hence, the conditional distribution of $\mu$ given $x_1, \ldots, x_n$ is the normal distribution with mean

$$\frac{1}{n+1} \left( \sum_{i=1}^{n} x_i \right)$$

and variance

$$\frac{1}{n+1}$$

Thus, the Bayes' estimate of $\mu$ is given by

$$\hat{\mu}_B = E(\mu \mid x_1, \ldots, x_n) = \frac{1}{n+1} \sum_{i=1}^{n} x_i \tag{7.49}$$

and the Bayes' estimator of $\mu$ is

$$M_B = \frac{1}{n+1}\sum_{i=1}^{n} X_i = \frac{n}{n+1}\bar{X} \tag{7.50}$$

**7.14.** Let $(X_1, \ldots, X_n)$ be a random sample of a r.v. $X$ with pdf $f(x; \theta)$, where $\theta$ is an unknown parameter. The statistics $L$ and $U$ determine a $100(1-\alpha)$ percent *confidence interval* $(L, U)$ for the parameter $\theta$ if

$$P(L < \theta < U) \geq 1 - \alpha \qquad 0 < \alpha < 1 \tag{7.51}$$

and $1 - \alpha$ is called the *confidence coefficient*. Find $L$ and $U$ if $X$ is a normal r.v. with known variance $\sigma^2$ and mean $\mu$ is an unknown parameter.

If $X = N(\mu; \sigma^2)$, then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \qquad \text{where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

is a standard normal r.v., and hence for a given $\alpha$ we can find a number $z_{\alpha/2}$ from Table A (Appendix A) such that

$$P\left(-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = 1 - \alpha \tag{7.52}$$

For example, if $1 - \alpha = 0.95$, then $z_{\alpha/2} = z_{0.025} = 1.96$, and if $1 - \alpha = 0.9$, then $z_{\alpha/2} = z_{0.05} = 1.645$. Now, recalling that $\sigma > 0$, we have the following equivalent inequality relationships;

$$-z_{\alpha/2} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < z_{\alpha/2}$$

$$-z_{\alpha/2}(\sigma/\sqrt{n}) < \bar{X} - \mu < z_{\alpha/2}(\sigma/\sqrt{n})$$

$$-\bar{X} - z_{\alpha/2}(\sigma/\sqrt{n}) < -\mu < -\bar{X} + z_{\alpha/2}(\sigma/\sqrt{n})$$

and

$$\bar{X} + z_{\alpha/2}(\sigma/\sqrt{n}) > \mu > \bar{X} - z_{\alpha/2}(\sigma/\sqrt{n})$$

Thus, we have

$$P[\bar{X} - z_{\alpha/2}(\sigma/\sqrt{n}) < \mu < \bar{X} + z_{\alpha/2}(\sigma/\sqrt{n})] = 1 - \alpha \tag{7.53}$$

and so

$$L = \bar{X} - z_{\alpha/2}(\sigma/\sqrt{n}) \qquad \text{and} \qquad U = \bar{X} + z_{\alpha/2}(\sigma/\sqrt{n}) \tag{7.54}$$

**7.15.** Consider a normal r.v. with variance 1.66 and unknown mean $\mu$. Find the 95 percent confidence interval for the mean based on a random sample of size 10.

As shown in Prob. 7.14, for $1 - \alpha = 0.95$, we have $z_{\alpha/2} = z_{0.025} = 1.96$ and

$$z_{\alpha/2}(\sigma/\sqrt{n}) = 1.96(\sqrt{1.66}/\sqrt{10}) = 0.8$$

Thus, by Eq. (7.54), the 95 percent confidence interval for $\mu$ is

$$(\bar{X} - 0.8,\ \bar{X} + 0.8)$$

## MEAN SQUARE ESTIMATION

**7.16.** Find the m.s. estimate of a r.v. $Y$ by a constant $c$.

By Eq. (7.17), the m.s. error is

$$e = E[(Y - c)^2] = \int_{-\infty}^{\infty} (y - c)^2 f(y)\, dy \tag{7.55}$$

Clearly the m.s. error $e$ depends on $c$, and it is minimum if

$$\frac{de}{dc} = -2\int_{-\infty}^{\infty}(y-c)f(y)\,dy = 0$$

or

$$c\int_{-\infty}^{\infty} f(y)\,dy = c = \int_{-\infty}^{\infty} yf(y)\,dy$$

Thus, we conclude that the m.s. estimate $c$ of $Y$ is given by

$$\hat{y} = c = \int_{-\infty}^{\infty} yf(y)\,dy = E(Y) \tag{7.56}$$

**7.17.** Find the m.s. estimator of a r.v. $Y$ by a function $g(X)$ of the r.v. $X$.

By Eq. (7.17), the m.s. error is

$$e = E\{[Y-g(X)]^2\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[y-g(x)]^2 f(x,y)\,dx\,dy$$

Since $f(x, y) = f(y\,|\,x)f(x)$, we can write

$$e = \int_{-\infty}^{\infty} f(x)\left\{\int_{-\infty}^{\infty}[y-g(x)]^2 f(y\,|\,x)\,dy\right\}dx \tag{7.57}$$

Since the integrands above are positive, the m.s. error $e$ is minimum if the inner integrand,

$$\int_{-\infty}^{\infty}[y-g(x)]^2 f(y\,|\,x)\,dy \tag{7.58}$$

is minimum for every $x$. Comparing Eq. (7.58) with Eq. (7.55) (Prob. 7.16), we see that they are the same form if $c$ is changed to $g(x)$ and $f(y)$ is changed to $f(y\,|\,x)$. Thus, by the result of Prob. 7.16 [Eq. (7.56)], we conclude that the m.s. estimate of $Y$ is given by

$$\hat{y} = g(x) = \int_{-\infty}^{\infty} yf(y\,|\,x)\,dy = E(Y\,|\,x) \tag{7.59}$$

Hence, the m.s. estimator of $Y$ is

$$\hat{Y} = g(X) = E(Y\,|\,X) \tag{7.60}$$

**7.18.** Find the m.s. error if $g(x) = E(Y\,|\,x)$ is the m.s. estimate of $Y$.

As we see from Eq. (3.58), the conditional mean $E(Y\,|\,x)$ of $Y$, given that $X = x$, is a function of $x$, and by Eq. (4.39),

$$E[E(Y\,|\,X)] = E(Y) \tag{7.61}$$

Similarly, the conditional mean $E[g(X, Y)\,|\,x]$ of $g(X, Y)$, given that $X = x$, is a function of $x$. It defines, therefore, the function $E[g(X, Y)\,|\,X]$ of the r.v. $X$. Then

$$\begin{aligned} E\{E[g(X,Y)\,|\,X]\} &= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(x,y)f(y\,|\,x)\,dy\right]f(x)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(y\,|\,x)f(x)\,dx\,dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)f(x,y)\,dx\,dy = E[g(X,Y)] \end{aligned} \tag{7.62}$$

Note that Eq. (7.62) is the generalization of Eq. (7.61). Next, we note that

$$E[g_1(X)g_2(Y)\,|\,x] = E[g_1(x)g_2(Y)\,|\,x] = g_1(x)E[g_2(Y)\,|\,x] \tag{7.63}$$

Then by Eqs. (7.62) and (7.63), we have

$$E[g_1(X)g_2(Y)] = E\{E[g_1(X)g_2(Y) \mid X]\} = E\{g_1(X)E(g_2(Y) \mid X]\} \tag{7.64}$$

Now, setting $g_1(X) = g(X)$ and $g_2(Y) = Y$ in Eq. (7.64), and using Eq. (7.18), we obtain

$$E[g(X)Y] = E[g(X)E(Y \mid X)] = E[g^2(X)]$$

Thus, the m.s. error is given by

$$\begin{aligned} e = E\{[Y - g(X)]^2\} &= E(Y^2) - 2E[g(X)Y] + E[g^2(X)] \\ &= E(Y^2) - E[g^2(X)] \end{aligned} \tag{7.65}$$

**7.19.** Let $Y = X^2$ and $X$ be a uniform r.v. over $(-1, 1)$. Find the m.s. estimator of $Y$ in terms of $X$ and its m.s. error.

By Eq. (7.18), the m.s. estimate of $Y$ is given by

$$g(x) = E(Y \mid x) = E(X^2 \mid X = x) = x^2$$

Hence, the m.s. estimator of $Y$ is

$$\hat{Y} = X^2 \tag{7.66}$$

The m.s. error is

$$e = E\{[Y - g(X)]^2\} = E\{[X^2 - X^2]^2\} = 0 \tag{7.67}$$

## LINEAR MEAN SQUARE ESTIMATION

**7.20.** Derive the orthogonality principle (7.21) and Eq. (7.22).

By Eq. (7.20), the m.s. error is

$$e(a, b) = E\{[Y - (aX + b)]^2\}$$

Clearly, the m.s. error $e$ is a function of $a$ and $b$, and it is minimum if $\partial e/\partial a = 0$ and $\partial e/\partial b = 0$. Now

$$\frac{\partial e}{\partial a} = E\{2[Y - (aX + b)](-X)\} = -2E\{[Y - (aX + b)]X\}$$

$$\frac{\partial e}{\partial b} = E\{2[Y - (aX + b)](-1)\} = -2E\{[Y - (aX + b)]\}$$

Setting $\partial e/\partial a = 0$ and $\partial e/\partial b = 0$, we obtain

$$E\{[Y - (aX + b)]X\} = 0 \tag{7.68}$$

$$E[Y - (aX + b)] = 0 \tag{7.69}$$

Note that Eq. (7.68) is the orthogonality principle (7.21).

Rearranging Eqs. (7.68) and (7.69), we get

$$E(X^2)a + E(X)b = E(XY)$$

$$E(X)a + b = E(Y)$$

Solving for $a$ and $b$, we obtain Eq. (7.22); that is,

$$a = \frac{E(XY) - E(X)E(Y)}{E(X^2) - [E(X)]^2} = \frac{\sigma_{XY}}{\sigma_X^2} = \frac{\sigma_Y}{\sigma_X}\rho_{XY}$$

$$b = E(Y) - aE(X) = \mu_Y - a\mu_X$$

where we have used Eqs. (2.31), (3.51), and (3.53).

**7.21.** Show that m.s. error defined by Eq. (7.20) is minimum when Eqs. (7.68) and (7.69) are satisfied.

Assume that $\hat{Y} = cX + d$, where $c$ and $d$ are arbitrary constants. Then

$$\begin{aligned} e(c, d) &= E\{[Y - (cX + d)]^2\} = E\{[Y - (aX + b) + (a - c)X + (b - d)]^2\} \\ &= E\{[Y - (aX + b)]^2\} + E\{[(a - c)X + (b - d)]^2\} \\ &\quad + 2(a - c)E\{[Y - (aX + b)]X\} + 2(b - d)E\{[Y - (aX + b)]\} \\ &= e(a, b) + E\{[(a - c)X + (b - d)]^2\} \\ &\quad + 2(a - c)E\{[Y - (aX + b)]X\} + 2(b - d)E\{[Y - (aX + b)]\} \end{aligned}$$

The last two terms on the right-hand side are zero when Eqs. (*7.68*) and (*7.69*) are satisfied, and the second term on the right-hand side is positive if $a \neq c$ and $b \neq d$. Thus, $e(c, d) \geq e(a, b)$ for any $c$ and $d$. Hence, $e(a, b)$ is minimum.

**7.22.** Derive Eq. (*7.23*).

By Eqs. (*7.68*) and (*7.69*), we have

$$R\{[Y - (aX + b)]aX\} = 0 = E\{[Y - (aX + b)]b\}$$

Then
$$\begin{aligned} e_m = e(a, b) &= E\{[Y - (aX + b)]^2\} = E\{[Y - (aX + b)][Y - (aX + b)]\} \\ &= E\{[Y - (aX + b)]Y\} = E(Y^2) - aE(XY) - bE(Y) \end{aligned}$$

Using Eqs. (*2.31*), (*3.51*), and (*3.53*), and substituting the values of $a$ and $b$ [Eq. (*7.22*)] in the above expression, the minimum m.s. error is

$$\begin{aligned} e_m &= \sigma_Y^2 + \mu_Y^2 - a(\sigma_{XY} + \mu_X\mu_Y) - (\mu_Y - a\mu_X)\mu_Y \\ &= \sigma_Y^2 - a\sigma_{XY} = \sigma_Y^2 - \frac{\sigma_{XY}^2}{\sigma_X^2} = \sigma_Y^2\left(1 - \frac{\sigma_{XY}^2}{\sigma_X^2\sigma_Y^2}\right) = \sigma_Y^2(1 - \rho_{XY}^2) \end{aligned}$$

which is Eq. (*7.23*).

**7.23.** Let $Y = X^2$, and let $X$ be a uniform r.v. over $(-1, 1)$ (see Prob. 7.19). Find the linear m.s. estimator of $Y$ in terms of $X$ and its m.s. error.

The linear m.s. estimator of $Y$ in terms of $X$ is

$$\hat{Y} = aX + b$$

where $a$ and $b$ are given by [Eq. (*7.22*)]

$$a = \frac{\sigma_{XY}}{\sigma_X^2} \qquad b = \mu_Y - a\mu_X$$

Now, by Eqs. (*2.46*) and (*2.44*),

$$\mu_X = E(X) = 0$$

$$E(XY) = E(XX^2) = E(X^3) = \frac{1}{2}\int_{-1}^{1} x^3\,dx = 0$$

By Eq. (*3.51*),

$$\sigma_{XY} = \mathrm{Cov}(XY) = E(XY) - E(X)E(Y) = 0$$

Thus, $a = 0$ and $b = E(Y)$, and the linear m.s. estimator of $Y$ is

$$\hat{Y} = b = E(Y) \tag{7.70}$$

and the m.s. error is

$$e = E\{[Y - E(Y)]^2\} = \sigma_Y^2 \tag{7.71}$$

**7.24.** Find the minimum m.s. error estimator of $Y$ in terms of $X$ when $X$ and $Y$ are jointly normal r.v.'s.

By Eq. (*7.18*), the minimum m.s. error estimator of $Y$ in terms of $X$ is

$$\hat{Y} = E(Y \mid X)$$

Now, when $X$ and $Y$ are jointly normal, by Eq. (*3.108*) (Prob. 3.51), we have

$$E(Y \mid x) = \rho_{XY} \frac{\sigma_Y}{\sigma_X} X + \mu_Y - \rho_{XY} \frac{\sigma_Y}{\sigma_X} \mu_X$$

Hence, the minimum m.s. error estimator of $Y$ is

$$\hat{Y} = E(Y \mid X) = \rho_{XY} \frac{\sigma_Y}{\sigma_X} X + \mu_Y - \rho_{XY} \frac{\sigma_Y}{\sigma_X} \mu_X \tag{7.72}$$

Comparing Eq. (*7.72*) with Eqs. (*7.19*) and (*7.22*), we see that for jointly normal r.v.'s the linear m.s. estimator is the minimum m.s. error estimator.

## Supplementary Problems

**7.25.** Let $(X_1, \ldots, X_n)$ be a random sample of $X$ having unknown mean $\mu$ and variance $\sigma^2$. Show that the estimator of $\sigma^2$ defined by

$$S_1^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

where $\bar{X}$ is the sample mean, is an unbiased estimator of $\sigma^2$. Note that $S_1^2$ is often called the *sample variance*.

*Hint:* Show that $S_1^2 = \dfrac{n}{n-1} S^2$, and use Eq. (*7.29*).

**7.26.** Let $(X_1, \ldots, X_n)$ be a random sample of $X$ having known mean $\mu$ and unknown variance $\sigma^2$. Show that the estimator of $\sigma^2$ defined by

$$S_0^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$$

is an unbiased estimator of $\sigma^2$.

*Hint:* Proceed as in Prob. 7.2.

**7.27.** Let $(X_1, \ldots, X_n)$ be a random sample of a binomial r.v. $X$ with parameter $(m, p)$, where $p$ is unknown. Show that the maximum-likelihood estimator of $p$ given by Eq. (*7.34*) is unbiased.

*Hint:* Use Eq. (*2.38*).

**7.28.** Let $(X_1, \ldots, X_n)$ be a random sample of a Bernoulli r.v. $X$ with pmf $f(x; p) = p^x(1-p)^{1-x}$, $x = 0, 1$, where $p$, $0 \le p \le 1$, is unknown. Find the maximum-likelihood estimator of $p$.

*Ans.* $P_{ML} = \dfrac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$

**7.29.** The values of a random sample, 2.9, 0.5, 1.7, 4.3, and 3.2, are obtained from a r.v. $X$ that is uniformly distributed over the unknown interval $(a, b)$. Find the maximum-likelihood estimates of $a$ and $b$.

*Ans.* $\hat{a}_{ML} = \min_i x_i = 0.5, \qquad \hat{b}_{ML} = \max_i x_i = 4.3$

**7.30.** In analyzing the flow of traffic through a drive-in bank, the times (in minutes) between arrivals of 10 customers are recorded as 3.2, 2.1, 5.3, 4.2, 1.2, 2.8, 6.4, 1.5, 1.9, and 3.0. Assuming that the interarrival time is an exponential r.v. with parameter $\lambda$, find the maximum likelihood estimate of $\lambda$.

*Ans.* $\hat{\lambda}_{ML} = \dfrac{1}{3.16}$

**7.31.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. $X$ with known mean $\mu$ and unknown variance $\sigma^2$. Find the maximum likelihood estimator of $\sigma^2$.

*Ans.* $S_{ML}{}^2 = \dfrac{1}{n}\sum_{i=1}^{n}(X_i - \mu)^2$

**7.32.** Let $(X_1, \ldots, X_n)$ be the random sample of a normal r.v. $X$ with mean $\mu$ and variance $\sigma^2$, where $\mu$ is unknown. Assume that $\mu$ is itself to be a normal r.v. with mean $\mu_1$ and variance $\sigma_1{}^2$. Find the Bayes' estimate of $\mu$.

*Ans.* $\hat{\mu}_B = \left(\dfrac{\mu_1}{\sigma_1{}^2} + \dfrac{n\bar{x}}{\sigma^2}\right) \Big/ \left(\dfrac{1}{\sigma_1{}^2} + \dfrac{n}{\sigma^2}\right) \qquad \bar{x} = \dfrac{1}{n}\sum_{i=1}^{n} x_i$

**7.33.** Let $(X_1, \ldots, X_n)$ be the random sample of a normal r.v. $X$ with variance 100 and unknown $\mu$. What sample size $n$ is required such that the width of 95 percent confidence interval is 5?

*Ans.* $n = 62$

**7.34.** Find a constant $a$ such that if $Y$ is estimated by $aX$, the m.s. error is minimum, and also find the minimum m.s. error $e_m$.

*Ans.* $a = E(XY)/E(X^2) \qquad e_m = E(Y^2) - [E(XY)]^2/[E(X)]^2$

**7.35.** Derive Eqs. (7.25) and (7.26).

*Hint:* Proceed as in Prob. 7.20.

# Chapter 8

# Decision Theory

## 8.1 INTRODUCTION

There are many situations in which we have to make decisions based on observations or data that are random variables. The theory behind the solutions for these situations is known as *decision theory* or *hypothesis testing*. In communication or radar technology, decision theory or hypothesis testing is known as (signal) detection theory. In this chapter we present a brief review of the binary decision theory and various decision tests.

## 8.2 HYPOTHESIS TESTING

### A. Definitions:

A *statistical hypothesis* is an assumption about the probability law of r.v.'s. Suppose we observe a random sample $(X_1, \ldots, X_n)$ of a r.v. $X$ whose pdf $f(\mathbf{x}; \theta) = f(x_1, \ldots, x_n; \theta)$ depends on a parameter $\theta$. We wish to test the assumption $\theta = \theta_0$ against the assumption $\theta = \theta_1$. The assumption $\theta = \theta_0$ is denoted by $H_0$ and is called the *null hypothesis*. The assumption $\theta = \theta_1$ is denoted by $H_1$ and is called the *alternative hypothesis*.

$$H_0: \quad \theta = \theta_0 \quad \text{(Null hypothesis)}$$
$$H_1: \quad \theta = \theta_1 \quad \text{(Alternative hypothesis)}$$

A hypothesis is called *simple* if all parameters are specified exactly. Otherwise it is called *composite*. Thus, suppose $H_0: \theta = \theta_0$ and $H_1: \theta \neq \theta_0$; then $H_0$ is simple and $H_1$ is composite.

### B. Hypothesis Testing and Types of Errors:

Hypothesis testing is a decision process establishing the validity of a hypothesis. We can think of the decision process as dividing the observation space $R^n$ (*Euclidean n-space*) into two regions $R_0$ and $R_1$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the observed vector. Then if $\mathbf{x} \in R_0$, we will decide on $H_0$; if $\mathbf{x} \in R_1$, we decide on $H_1$. The region $R_0$ is known as the *acceptance region* and the region $R_1$ as the *rejection* (or *critical*) *region* (since the null hypothesis is rejected). Thus, with the observation vector (or data), one of the following four actions can happen:

1. $H_0$ true; accept $H_0$
2. $H_0$ true; reject $H_0$ (or accept $H_1$)
3. $H_1$ true; accept $H_1$
4. $H_1$ true; reject $H_1$ (or accept $H_0$)

The first and third actions correspond to correct decisions, and the second and fourth actions correspond to errors. The errors are classified as

1. Type I error: Reject $H_0$ (or accept $H_1$) when $H_0$ is true.
2. Type II error: Reject $H_1$ (or accept $H_0$) when $H_1$ is true.

Let $P_{\mathrm{I}}$ and $P_{\mathrm{II}}$ denote, respectively, the probabilities of Type I and Type II errors:

$$P_{\mathrm{I}} = P(D_1 \mid H_0) = P(\mathbf{x} \in R_1; H_0) \tag{8.1}$$

$$P_{\mathrm{II}} = P(D_0 \mid H_1) = P(\mathbf{x} \in R_0; H_1) \tag{8.2}$$

where $D_i$ ($i = 0, 1$) denotes the event that the decision is made to accept $H_i$. $P_{\text{I}}$ is often denoted by $\alpha$ and is known as the *level of significance*, and $P_{\text{II}}$ is denoted by $\beta$ and $(1 - \beta)$ is known as the *power of the test*. Note that since $\alpha$ and $\beta$ represent probabilities of events from the same decision problem, they are not independent of each other or of the sample size $n$. It would be desirable to have a decision process such that both $\alpha$ and $\beta$ will be small. However, in general, a decrease in one type of error leads to an increase in the other type for a fixed sample size (Prob. 8.4). The only way to simultaneously reduce both type of errors is to increase the sample size (Prob. 8.5). One might also attach some relative importance (or cost) to the four possible courses of action and minimize the total cost of the decision (see Sec. 8.3D).

The probabilities of correct decisions (actions 1 and 3) may be expressed as

$$P(D_0 \mid H_0) = P(\mathbf{x} \in R_0; H_0) \tag{8.3}$$

$$P(D_1 \mid H_1) = P(\mathbf{x} \in R_1; H_1) \tag{8.4}$$

In radar signal detection, the two hypotheses are

$H_0$: No target exists

$H_1$: Target is present

In this case, the probability of a Type I error $P_{\text{I}} = P(D_1 \mid H_0)$ is often referred to as the *false-alarm* probability (denoted by $P_F$), the probability of a Type II error $P_{\text{II}} = P(D_0 \mid H_1)$ as the *miss* probability (denoted by $P_M$), and $P(D_1 \mid H_1)$ as the *detection* probability (denoted by $P_D$). The cost of failing to detect a target cannot be easily determined. In general we set a value of $P_F$ which is acceptable and seek a decision test that constrains $P_F$ to this value while maximizing $P_D$ (or equivalently minimizing $P_M$). This test is known as the *Neyman-Pearson* test (see Sec. 8.3C).

## 8.3 DECISION TESTS

### A. Maximum-Likelihood Test:

Let $\mathbf{x}$ be the observation vector and $P(\mathbf{x} \mid H_i)$, $i = 0.1$, denote the probability of observing $\mathbf{x}$ given that $H_i$ was true. In the *maximum-likelihood* test, the decision regions $R_0$ and $R_1$ are selected as

$$\begin{aligned} R_0 &= \{\mathbf{x}: P(\mathbf{x} \mid H_0) > P(\mathbf{x} \mid H_1)\} \\ R_1 &= \{\mathbf{x}: P(\mathbf{x} \mid H_0) < P(\mathbf{x} \mid H_1)\} \end{aligned} \tag{8.5}$$

Thus, the maximum-likelihood test can be expressed as

$$d(\mathbf{x}) = \begin{cases} H_0 & \text{if } P(\mathbf{x} \mid H_0) > P(\mathbf{x} \mid H_1) \\ H_1 & \text{if } P(\mathbf{x} \mid H_0) < P(\mathbf{x} \mid H_1) \end{cases} \tag{8.6}$$

The above decision test can be rewritten as

$$\frac{P(\mathbf{x} \mid H_1)}{P(\mathbf{x} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{8.7}$$

If we define the likelihood ratio $\Lambda(\mathbf{x})$ as

$$\Lambda(\mathbf{x}) = \frac{P(\mathbf{x} \mid H_1)}{P(\mathbf{x} \mid H_0)} \tag{8.8}$$

then the maximum-likelihood test (8.7) can be expressed as

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{8.9}$$

which is called the *likelihood ratio test*, and 1 is called the *threshold value* of the test.

Note that the likelihood ratio $\Lambda(\mathbf{x})$ is also often expressed as

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} \tag{8.10}$$

**B. MAP Test:**

Let $P(H_i \mid \mathbf{x})$, $i = 0, 1$, denote the probability that $H_i$ was true given a particular value of $\mathbf{x}$. The conditional probability $P(H_i \mid \mathbf{x})$ is called *a posteriori* (or posterior) probability, that is, a probability that is computed after an observation has been made. The probability $P(H_i)$, $i = 0, 1$, is called *a priori* (or prior) probability. In the *maximum a posteriori* (MAP) test, the decision regions $R_0$ and $R_1$ are selected as

$$\begin{aligned} R_0 &= \{\mathbf{x}: P(H_0 \mid \mathbf{x}) > P(H_1 \mid \mathbf{x})\} \\ R_1 &= \{\mathbf{x}: P(H_0 \mid \mathbf{x}) < P(H_1 \mid \mathbf{x})\} \end{aligned} \tag{8.11}$$

Thus, the MAP test is given by

$$d(\mathbf{x}) = \begin{cases} H_0 & \text{if } P(H_0 \mid \mathbf{x}) > P(H_1 \mid \mathbf{x}) \\ H_1 & \text{if } P(H_0 \mid \mathbf{x}) < P(H_1 \mid \mathbf{x}) \end{cases} \tag{8.12}$$

which can be rewritten as

$$\frac{P(H_1 \mid \mathbf{x})}{P(H_0 \mid \mathbf{x})} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{8.13}$$

Using Bayes' rule [Eq. (*1.42*)], Eq. (*8.13*) reduces to

$$\frac{P(\mathbf{x} \mid H_1)P(H_1)}{P(\mathbf{x} \mid H_0)P(H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \tag{8.14}$$

Using the likelihood ratio $\Lambda(\mathbf{x})$ defined in Eq. (*8.8*), the MAP test can be expressed in the following likelihood ratio test as

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{P(H_0)}{P(H_1)} \tag{8.15}$$

where $\eta = P(H_0)/P(H_1)$ is the threshold value for the MAP test. Note that when $P(H_0) = P(H_1)$, the maximum-likelihood test is also the MAP test.

**C. Neyman-Pearson Test:**

As we mentioned before, it is not possible to simultaneously minimize both $\alpha(=P_{\mathrm{I}})$ and $\beta(=P_{\mathrm{II}})$. The Neyman-Pearson test provides a workable solution to this problem in that the test minimizes $\beta$ for a given level of $\alpha$. Hence, the Neyman-Pearson test is the test which maximizes the power of the test $1 - \beta$ for a given level of significance $\alpha$. In the Neyman-Pearson test, the critical (or rejection) region $R_1$ is selected such that $1 - \beta = 1 - P(D_0 \mid H_1) = P(D_1 \mid H_1)$ is maximum subject to the constraint $\alpha = P(D_1 \mid H_0) = \alpha_0$. This is a classical problem in optimization: maximizing a function subject to a constraint, which can be solved by the use of Lagrange multiplier method. We thus construct the objective function

$$J = (1 - \beta) - \lambda(\alpha - \alpha_0) \tag{8.16}$$

where $\lambda \geq 0$ is a Lagrange multiplier. Then the critical region $R_1$ is chosen to maximize $J$. It can be shown that the Neyman-Pearson test can be expressed in terms of the likelihood ratio test as

(Prob. 8.8)

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \lambda \tag{8.17}$$

where the threshold value $\eta$ of the test is equal to the Lagrange multiplier $\lambda$, which is chosen to satisfy the contraint $\alpha = \alpha_0$.

**D. Bayes' Test:**

Let $C_{ij}$ be the cost associated with $(D_i, H_j)$, which denotes the event that we accept $H_i$ when $H_j$ is true. Then the average cost, which is known as the *Bayes' risk*, can be written as

$$\bar{C} = C_{00}P(D_0, H_0) + C_{10}P(D_1, H_0) + C_{01}P(D_0, H_1) + C_{11}P(D_1, H_1) \tag{8.18}$$

where $P(D_i, H_j)$ denotes the probability that we accept $H_i$ when $H_j$ is true. By Bayes' rule (*1.42*), we have

$$\bar{C} = C_{00}P(D_0 \mid H_0)P(H_0) + C_{10}P(D_1 \mid H_0)P(H_0) + C_{01}P(D_0 \mid H_1)P(H_1) + C_{11}P(D_1 \mid H_1)P(H_1) \tag{8.19}$$

In general, we assume that

$$C_{10} > C_{00} \qquad \text{and} \qquad C_{01} > C_{11} \tag{8.20}$$

since it is reasonable to assume that the cost of making an incorrect decision is higher than the cost of making a correct decision. The test that minimizes the average cost $\bar{C}$ is called the Bayes' test, and it can be expressed in terms of the likelihood ratio test as (Prob. 8.10)

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{(C_{10} - C_{00})P(H_0)}{(C_{01} - C_{11})P(H_1)} \tag{8.21}$$

Note that when $C_{10} - C_{00} = C_{01} - C_{11}$, the Bayes' test (*8.21*) and the MAP test (*8.15*) are identical.

**E. Minimum Probability of Error Test:**

If we set $C_{00} = C_{11} = 0$ and $C_{01} = C_{10} = 1$ in Eq. (*8.18*), we have

$$\bar{C} = P(D_1, H_0) + P(D_0, H_1) = P_e \tag{8.22}$$

which is just the probability of making an incorrect decision. Thus, in this case, the Bayes' test yields the minimum probability of error, and Eq. (*8.21*) becomes

$$\Lambda(\mathbf{x}) \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{P(H_0)}{P(H_1)} \tag{8.23}$$

We see that the minimum probability of error test is the same as the MAP test.

**F. Minimax Test:**

We have seen that the Bayes' test requires the a priori probabilities $P(H_0)$ and $P(H_1)$. Frequently, these probabilities are not known. In such a case, the Bayes' test cannot be applied, and the following minimax (min-max) test may be used. In the minimax test, we use the Bayes' test which corresponds to the least favorable $P(H_0)$ (Prob. 8.12). In the minimax test, the critical region $R_1^*$ is defined by

$$\max_{P(H_0)} \bar{C}[P(H_0), R_1^*] = \min_{R_1} \max_{P(H_0)} \bar{C}[P(H_0), R_1] < \max_{P(H_0)} \bar{C}[P(H_0), R_1] \tag{8.24}$$

for all $R_1 \neq R_1^*$. In other words, $R_1^*$ is the critical region which yields the minimum Bayes' risk for the least favorable $P(H_0)$. Assuming that the minimization and maximization operations are interchangeable, then we have

$$\min_{R_1} \max_{P(H_0)} \bar{C}[P(H_0), R_1] = \max_{P(H_0)} \min_{R_1} \bar{C}[P(H_0), R_1] \tag{8.25}$$

The minimization of $\bar{C}[P(H_0), R_1]$ with respect to $R_1$ is simply the Bayes' test, so that

$$\min_{R_1} \bar{C}[P(H_0), R_1] = C^*[P(H_0)] \tag{8.26}$$

where $C^*[P(H_0)]$ is the minimum Bayes' risk associated with the a priori probability $P(H_0)$. Thus, Eq. (8.25) states that we may find the minimax test by finding the Bayes' test for the least favorable $P(H_0)$, that is, the $P(H_0)$ which maximizes $\bar{C}[P(H_0)]$.

# Solved Problems

## HYPOTHESIS TESTING

**8.1.** Suppose a manufacturer of memory chips observes that the probability of chip failure is $p = 0.05$. A new procedure is introduced to improve the design of chips. To test this new procedure, 200 chips could be produced using this new procedure and tested. Let r.v. $X$ denote the number of these 200 chips that fail. We set the test rule that we would accept the new procedure if $X \leq 5$. Let

$$H_0: \quad p = 0.05 \qquad \text{(No change hypothesis)}$$
$$H_1: \quad p < 0.05 \qquad \text{(Improvement hypothesis)}$$

Find the probability of a Type I error.

If we assume that these tests using the new procedure are independent and have the same probability of failure on each test, then $X$ is a binomial r.v. with parameters $(n, p) = (200, p)$. We make a Type I error if $X \leq 5$ when in fact $p = 0.05$. Thus, using Eq. (2.37), we have

$$P_1 = P(D_1 \mid H_0) = P(X \leq 5;\ p = 0.05)$$
$$= \sum_{k=0}^{5} \binom{200}{k} (0.05)^k (0.95)^{200-k}$$

Since $n$ is rather large and $p$ is small, these binomial probabilities can be approximated by Poisson probabilities with $\lambda = np = 200(0.05) = 10$ (see Prob. 2.40). Thus, using Eq. (2.100), we obtain

$$P_1 \approx \sum_{k=0}^{5} e^{-10} \frac{10^k}{k!} = 0.067$$

Note that $H_0$ is a simple hypothesis but $H_1$ is a composite hypothesis.

**8.2.** Consider again the memory chip manufacturing problem of Prob. 8.1. Now let

$$H_0: \quad p = 0.05 \qquad \text{(No change hypothesis)}$$
$$H_1: \quad p = 0.02 \qquad \text{(Improvement hypothesis)}$$

Again our rule is, we would reject the new procedure if $X > 5$. Find the probability of a Type II error.

Now both hypotheses are simple. We make a Type II error if $X > 5$ when in fact $p = 0.02$. Hence, by Eq. (2.37),

$$P_{II} = P(D_0 \mid H_1) = P(X > 5; p = 0.02)$$
$$= \sum_{k=6}^{\infty} \binom{200}{k} (0.02)^k (0.98)^{200-k}$$

Again using the Poisson approximation with $\lambda = np = 200(0.02) = 4$, we obtain

$$P_{II} \approx 1 - \sum_{k=0}^{5} e^{-4} \frac{4^k}{k!} = 0.215$$

**8.3.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. $X$ with mean $\mu$ and variance 100. Let

$$H_0: \quad \mu = 50$$
$$H_1: \quad \mu = \mu_1 \ (> 50)$$

and sample size $n = 25$. As a decision procedure, we use the rule to reject $H_0$ if $\bar{x} \geq 52$, where $\bar{x}$ is the value of the sample mean $\bar{X}$ defined by Eq. (7.27).

(*a*) Find the probability of rejecting $H_0$: $\mu = 50$ as a function of $\mu$ ($> 50$).

(*b*) Find the probability of a Type I error $\alpha$.

(*c*) Find the probability of a Type II error $\beta$ (i) when $\mu_1 = 53$ and (ii) when $\mu_1 = 55$.

(*a*) Since the test calls for the rejection of $H_0$: $\mu = 50$ when $\bar{x} \geq 52$, the probability of rejecting $H_0$ is given by

$$g(\mu) = P(\bar{X} \geq 52; \mu) \tag{8.27}$$

Now, by Eqs. (4.112) and (7.27), we have

$$\mathrm{Var}(\bar{X}) = \sigma_{\bar{X}}^2 = \frac{1}{n}\sigma^2 = \frac{100}{25} = 4$$

Thus, $\bar{X}$ is $N(\mu; 4)$, and using Eq. (2.55), we obtain

$$g(\mu) = P\left(\frac{\bar{X} - \mu}{2} \geq \frac{52 - \mu}{2}; \mu\right) = 1 - \Phi\left(\frac{52 - \mu}{2}\right) \qquad \mu \geq 50 \tag{8.28}$$

The function $g(\mu)$ is known as the *power function of the test*, and the value of $g(\mu)$ at $\mu = \mu_1$, $g(\mu_1)$, is called the *power* at $\mu_1$.

(*b*) Note that the power at $\mu = 50$, $g(50)$, is the probability of rejecting $H_0$: $\mu = 50$ when $H_0$ is true—that is, a Type I error. Thus, using Table A (Appendix A), we obtain

$$\alpha = P_I = g(50) = 1 - \Phi\left(\frac{52 - 50}{2}\right) = 1 - \Phi(1) = 0.1587$$

(*c*) Note that the power at $\mu = \mu_1$, $g(\mu_1)$, is the probability of rejecting $H_0$: $\mu = 50$ when $\mu = \mu_1$. Thus, $1 - g(\mu_1)$ is the probability of accepting $H_0$ when $\mu = \mu_1$—that is, the probability of a Type II error $\beta$.

(i) Setting $\mu = \mu_1 = 53$ in Eq. (8.28) and using Table A (Appendix A), we obtain

$$\beta = P_{II} = 1 - g(53) = \Phi\left(\frac{52 - 53}{2}\right) = \Phi(-\tfrac{1}{2}) = 1 - \Phi(\tfrac{1}{2}) = 0.3085$$

(ii) Similarly, for $\mu = \mu_1 = 55$ we obtain

$$\beta = P_{II} = 1 - g(55) = \Phi\left(\frac{52 - 55}{2}\right) = \Phi(-\tfrac{3}{2}) = 1 - \Phi(\tfrac{3}{2}) = 0.0668$$

Notice that clearly, the probability of a Type II error depends on the value of $\mu_1$.

**8.4.** Consider the binary decision problem of Prob. 8.3. We modify the decision rule such that we reject $H_0$ if $x \geq c$.

(*a*) Find the value of $c$ such that the probability of a Type I error $\alpha = 0.05$.

(*b*) Find the probability of a Type II error $\beta$ when $\mu_1 = 55$ with the modified decision rule.

(*a*) Using the result of part (*b*) in Prob. 8.3, $c$ is selected such that [see Eq. (*8.27*)]

$$\alpha = g(50) = P(\bar{X} \geq c;\ \mu = 50) = 0.05$$

However, when $\mu = 50$, $\bar{X} = N(50;\ 4)$, and [see Eq. (*8.28*)]

$$g(50) = P\left(\frac{\bar{X} - 50}{2} \geq \frac{c - 50}{2};\ \mu = 50\right) = 1 - \Phi\left(\frac{c - 50}{2}\right) = 0.05$$

From Table A (Appendix A), we have $\Phi(1.645) = 0.95$. Thus

$$\frac{c - 50}{2} = 1.645 \qquad \text{and} \qquad c = 50 + 2(1.645) = 53.29$$

(*b*) The power function $g(\mu)$ with the modified decision rule is

$$g(\mu) = P(\bar{X} \geq 53.29;\ \mu) = P\left(\frac{\bar{X} - \mu}{2} \geq \frac{53.29 - \mu}{2};\ \mu\right) = 1 - \Phi\left(\frac{53.29 - \mu}{2}\right)$$

Setting $\mu = \mu_1 = 55$ and using Table A (Appendix A), we obtain

$$\beta = P_{II} = 1 - g(55) = \Phi\left(\frac{53.29 - 55}{2}\right) = \Phi(-0.855)$$
$$= 1 - \Phi(0.855) = 0.1963$$

Comparing with the results of Prob. 8.3, we notice that with the change of the decision rule, $\alpha$ is reduced from 0.1587 to 0.05, but $\beta$ is increased from 0.0668 to 0.1963.

**8.5.** Redo Prob. 8.4 for the case where the sample size $n = 100$.

(*a*) With $n = 100$, we have

$$\text{Var}(\bar{X}) = \sigma_{\bar{X}}^2 = \frac{1}{n}\sigma^2 = \frac{100}{100} = 1$$

As in part (*a*) of Prob. 8.4, $c$ is selected so that

$$\alpha = g(50) = P(\bar{X} \geq c;\ \mu = 50) = 0.05$$

Since $\bar{X} = N(50;\ 1)$, we have

$$g(50) = P\left(\frac{\bar{X} - 50}{1} \geq \frac{c - 50}{1};\ \mu = 50\right) = 1 - \Phi(c - 50) = 0.05$$

Thus $$c - 50 = 1.645 \qquad \text{and} \qquad c = 51.645$$

(*b*) The power function is

$$g(\mu) = P(\bar{X} \geq 51.645;\ \mu)$$
$$= P\left(\frac{\bar{X} - \mu}{1} \geq \frac{51.645 - \mu}{1};\ \mu\right) = 1 - \Phi(51.645 - \mu)$$

Setting $\mu = \mu_1 = 55$ and using Table A (Appendix A), we obtain

$$\beta = P_{II} = 1 - g(55) = \Phi(51.645 - 55) = \Phi(-3.355) \approx 0.0004$$

Notice that with sample size $n = 100$, both $\alpha$ and $\beta$ have decreased from their respective original values of 0.1587 and 0.0668 when $n = 25$.

## DECISION TESTS

**8.6.** In a simple binary communication system, during every $T$ seconds, one of two possible signals $s_0(t)$ and $s_1(t)$ is transmitted. Our two hypotheses are

$$H_0: \quad s_0(t) \text{ was transmitted.}$$
$$H_1: \quad s_1(t) \text{ was transmitted.}$$

We assume that

$$s_0(t) = 0 \qquad \text{and} \qquad s_1(t) = 1 \qquad 0 < t < T$$

The communication channel adds noise $n(t)$, which is a zero-mean normal random process with variance 1. Let $x(t)$ represent the received signal:

$$x(t) = s_i(t) + n(t) \qquad i = 0, 1$$

We observe the received signal $x(t)$ at some instant during each signaling interval. Suppose that we received an observation $x = 0.6$.

(*a*) Using the maximum likelihood test, determine which signal is transmitted.

(*b*) Find $P_{\text{I}}$ and $P_{\text{II}}$.

(*a*) The received signal under each hypothesis can be written as

$$H_0: \quad x = n$$
$$H_1: \quad x = 1 + n$$

Then the pdf of $x$ under each hypothesis is given by

$$f(x \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$
$$f(x \mid H_1) = \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}$$

The likelihood ratio is then given by

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = e^{(x - 1/2)}$$

By Eq. (8.9), the maximum likelihood test is

$$e^{(x-1/2)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Taking the natural logarithm of the above expression, we get

$$x - \tfrac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} 0 \qquad \text{or} \qquad x \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2}$$

Since $x = 0.6 > \frac{1}{2}$, we determine that signal $s_1(t)$ was transmitted.

(*b*) The decision regions are given by

$$R_0 = \{x: x < \tfrac{1}{2}\} = (-\infty, \tfrac{1}{2}) \qquad R_1 = \{x: x > \tfrac{1}{2}\} = (\tfrac{1}{2}, \infty)$$

Then by Eqs. (8.1) and (8.2) and using Table A (Appendix A), we obtain

$$P_{\mathrm{I}} = P(D_1 \mid H_0) = \int_{R_1} f(x \mid H_0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-x^2/2}\, dx = 1 - \Phi(\tfrac{1}{2}) = 0.3085$$

$$P_{\mathrm{II}} = P(D_0 \mid H_1) = \int_{R_0} f(x \mid H_1)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1/2} e^{-(x-1)^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1/2} e^{-y^2/2}\, dy = \Phi(-\tfrac{1}{2}) = 0.3085$$

**8.7.** In the binary communication system of Prob. 8.6, suppose that $P(H_0) = \frac{2}{3}$ and $P(H_1) = \frac{1}{3}$.

(*a*) Using the MAP test, determine which signal is transmitted when $x = 0.6$.

(*b*) Find $P_{\mathrm{I}}$ and $P_{\mathrm{II}}$.

(*a*) Using the result of Prob. 8.6 and Eq. (8.15), the MAP test is given by

$$e^{(x-1/2)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)} = 2$$

Taking the natural logarithm of the above expression, we get

$$x - \tfrac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln 2 \qquad \text{or} \qquad x \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2} + \ln 2 = 1.193$$

Since $x = 0.6 < 1.193$, we determine that signal $s_0(t)$ was transmitted.

(*b*) The decision regions are given by

$$R_0 = \{x: x < 1.193\} = (-\infty, 1.193)$$
$$R_1 = \{x: x > 1.193\} = (1.193, \infty)$$

Thus, by Eqs. (8.1) and (8.2) and using Table A (Appendix A), we obtain

$$P_{\mathrm{I}} = P(D_1 \mid H_0) = \int_{R_1} f(x \mid H_0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{1.193}^{\infty} e^{-x^2/2}\, dx = 1 - \Phi(1.193) = 0.1164$$

$$P_{\mathrm{II}} = P(D_0 \mid H_1) = \int_{R_0} f(x \mid H_1)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{1.193} e^{-(x-1)^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.193} e^{-y^2/2}\, dy = \Phi(0.193) = 0.5765$$

**8.8.** Derive the Neyman-Pearson test, Eq. (8.17).

From Eq. (8.16), the objective function is

$$J = (1 - \beta) - \lambda(\alpha - \alpha_0) = P(D_1 \mid H_1) - \lambda[P(D_1 \mid H_0) - \alpha_0] \tag{8.29}$$

where $\lambda$ is an undetermined Lagrange multiplier which is chosen to satisfy the constraint $\alpha = \alpha_0$. Now, we wish to choose the critical region $R_1$ to maximize $J$. Using Eqs. (8.1) and (8.2), we have

$$J = \int_{R_1} f(\mathbf{x} \mid H_1)\, d\mathbf{x} - \lambda\left[\int_{R_1} f(\mathbf{x} \mid H_0)\, d\mathbf{x} - \alpha_0\right]$$

$$= \int_{R_1} [f(\mathbf{x} \mid H_1) - \lambda f(\mathbf{x} \mid H_0)]\, d\mathbf{x} + \lambda\alpha_0 \tag{8.30}$$

To maximize $J$ by selecting the critical region $R_1$, we select $\mathbf{x} \in R_1$ such that the integrand in Eq. (*8.30*) is positive. Thus $R_1$ is given by

$$R_1 = \{\mathbf{x}: [f(\mathbf{x} \mid H_1) - \lambda f(\mathbf{x} \mid H_0)] > 0\}$$

and the Neyman-Pearson test is given by

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

and $\lambda$ is determined such that the constraint

$$\alpha = P_I = P(D_1 \mid H_0) = \int_{R_1} f(\mathbf{x} \mid H_0)\, d\mathbf{x} = \alpha_0$$

is satisfied.

**8.9.** Consider the binary communication system of Prob. 8.6 and suppose that we require that $\alpha = P_I = 0.25$.

(*a*) Using the Neyman-Pearson test, determine which signal is transmitted when $x = 0.6$.

(*b*) Find $P_{II}$.

(*a*) Using the result of Prob. 8.6 and Eq. (*8.17*), the Neyman-Pearson test is given by

$$e^{(x - 1/2)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

Taking the natural logarithm of the above expression, we get

$$x - \tfrac{1}{2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \lambda \qquad \text{or} \qquad x \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2} + \ln \lambda$$

The critical region $R_1$ is thus

$$R_1 = \{x: x > \tfrac{1}{2} + \ln \lambda\}$$

Now we must determine $\lambda$ such that $\alpha = P_I = P(D_1 \mid H_0) = 0.25$. By Eq. (*8.1*), we have

$$P_I = P(D_1 \mid H_0) = \int_{R_1} f(x \mid H_0)\, dx = \frac{1}{\sqrt{2\pi}} \int_{1/2 + \ln \lambda}^{\infty} e^{-x^2/2}\, dx = 1 - \Phi\left(\frac{1}{2} + \ln \lambda\right)$$

Thus $$1 - \Phi(\tfrac{1}{2} + \ln \lambda) = 0.25 \qquad \text{or} \qquad \Phi(\tfrac{1}{2} + \ln \lambda) = 0.75$$

From Table A (Appendix A), we find that $\Phi(0.674) = 0.75$. Thus

$$\tfrac{1}{2} + \ln \lambda = 0.674 \rightarrow \lambda = 1.19$$

Then the Neyman-Pearson test is

$$x \underset{H_0}{\overset{H_1}{\gtrless}} 0.674$$

Since $x = 0.6 < 0.674$, we determine that signal $s_0(t)$ was transmitted.

(*b*) By Eq. (*8.2*), we have

$$P_{II} = P(D_0 \mid H_1) = \int_{R_0} f(x \mid H_1)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.674} e^{-(x - 1)^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.326} e^{-y^2/2}\, dy = \Phi(-0.326) = 0.3722$$

**8.10.** Derive Eq. (*8.21*).

By Eq. (*8.19*), the Bayes' risk is

$$\bar{C} = C_{00}\,P(D_0 \mid H_0)P(H_0) + C_{10}\,P(D_1 \mid H_0)P(H_0) + C_{01}P(D_0 \mid H_1)P(H_1) + C_{11}P(D_1 \mid H_1)P(H_1)$$

Now we can express

$$P(D_i \mid H_j) = \int_{R_i} f(\mathbf{x} \mid H_j)\, d\mathbf{x} \qquad i = 0, 1; j = 0, 1 \tag{8.31}$$

Then $\bar{C}$ can be expressed as

$$\begin{aligned}\bar{C} = C_{00}\,P(H_0) \int_{R_0} f(\mathbf{x} \mid H_0)\, d\mathbf{x} + C_{10}\,P(H_0) \int_{R_1} f(\mathbf{x} \mid H_0)\, d\mathbf{x} \\ + C_{01}P(H_1) \int_{R_0} f(\mathbf{x} \mid H_1)\, d\mathbf{x} + C_{11}P(H_1) \int_{R_1} f(\mathbf{x} \mid H_1)\, d\mathbf{x}\end{aligned} \tag{8.32}$$

Since $R_0 \cup R_1 = S$ and $R_0 \cap R_1 = \phi$, we can write

$$\int_{R_0} f(\mathbf{x} \mid H_j)\, d\mathbf{x} = \int_S f(\mathbf{x} \mid H_j)\, d\mathbf{x} - \int_{R_1} f(\mathbf{x} \mid H_j)\, d\mathbf{x} = 1 - \int_{R_1} f(\mathbf{x} \mid H_j)\, d\mathbf{x}$$

Then Eq. (*8.32*) becomes

$$\bar{C} = C_{00}\,P(H_0) + C_{01}P(H_1) + \int_{R_1} \{[(C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)] - [(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1)]\}\, d\mathbf{x}$$

The only variable in the above expression is the critical region $R_1$. By the assumptions [Eq. (*8.20*)] $C_{10} > C_{00}$ and $C_{01} > C_{11}$, the two terms inside the brackets in the integral are both positive. Thus, $\bar{C}$ is minimized if $R_1$ is chosen such that

$$(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1) > (C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)$$

for all $\mathbf{x} \in R_1$. That is, we decide to accept $H_1$ if

$$(C_{01} - C_{11})P(H_1)f(\mathbf{x} \mid H_1) > (C_{10} - C_{00})P(H_0)f(\mathbf{x} \mid H_0)$$

In terms of the likelihood ratio, we obtain

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})P(H_0)}{(C_{01} - C_{11})P(H_1)}$$

which is Eq. (*8.21*).

**8.11.** Consider a binary decision problem with the following conditional pdf's:

$$f(x \mid H_0) = \tfrac{1}{2}e^{-|x|}$$
$$f(x \mid H_1) = e^{-2|x|}$$

The Bayes' costs are given by

$$C_{00} = C_{11} = 0 \qquad C_{01} = 2 \qquad C_{10} = 1$$

(*a*) Determine the Bayes' test if $P(H_0) = \frac{2}{3}$ and the associated Bayes' risk.

(*b*) Repeat (*a*) with $P(H_0) = \frac{1}{2}$.

(*a*) The likelihood ratio is

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = \frac{e^{-2|x|}}{\frac{1}{2}e^{-|x|}} = 2e^{-|x|} \tag{8.33}$$

By Eq. (*8.21*), the Bayes' test is given by

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-0)\frac{2}{3}}{(2-0)\frac{1}{3}} = 1 \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{2}$$

Taking the natural logarithm of both sides of the last expression yields

$$|x| \underset{H_0}{\overset{H_1}{\lesssim}} -\ln(\tfrac{1}{2}) = 0.693$$

Thus, the decision regions are given by

$$R_0 = \{x: |x| > 0.693\} \qquad R_1 = \{x: |x| < 0.693\}$$

Then

$$P_I = P(D_1 \mid H_0) = \int_{-0.693}^{0.693} \tfrac{1}{2}e^{-|x|}\,dx = 2\int_0^{0.693} \tfrac{1}{2}e^{-x}\,dx = 0.5$$

$$P_{II} = P(D_0 \mid H_1) = \int_{-\infty}^{-0.693} e^{2x}\,dx + \int_{0.693}^{\infty} e^{-2x}\,dx = 2\int_{0.693}^{\infty} e^{-2x}\,dx = 0.25$$

and by Eq. (*8.19*), the Bayes' risk is

$$\bar{C} = P(D_1 \mid H_0)P(H_0) + 2P(D_0 \mid H_1)P(H_1) = (0.5)(\tfrac{2}{3}) + 2(0.25)(\tfrac{1}{3}) = 0.5$$

(*b*) The Bayes' test is

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(1-0)\frac{1}{2}}{(2-0)\frac{1}{2}} = \tfrac{1}{2} \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \tfrac{1}{4}$$

Again, taking the natural logarithm of both sides of the last expression yields

$$|x| \underset{H_0}{\overset{H_1}{\lesssim}} -\ln(\tfrac{1}{4}) = 1.386$$

Thus, the decision regions are given by

$$R_0 = \{x: |x| > 1.386\} \qquad R_1 = \{x: |x| < 1.386\}$$

Then

$$P_I = P(D_1 \mid H_0) = 2\int_0^{1.386} \tfrac{1}{2}e^{-x}\,dx = 0.75$$

$$P_{II} = P(D_0 \mid H_1) = 2\int_{1.386}^{\infty} e^{-2x}\,dx = 0.0625$$

and by Eq. (*8.19*), the Bayes' risk is

$$\bar{C} = (0.75)(\tfrac{1}{2}) + 2(0.0625)(\tfrac{1}{2}) = 0.4375$$

**8.12.** Consider the binary decision problem of Prob. 8.11 with the same Bayes' costs. Determine the minimax test.

From Eq. (*8.33*), the likelihood ratio is

$$\Lambda(x) = \frac{f(x \mid H_1)}{f(x \mid H_0)} = 2e^{-|x|}$$

In terms of $P(H_0)$, the Bayes' test [Eq. (*8.21*)] becomes

$$2e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\frac{P(H_0)}{1-P(H_0)} \qquad \text{or} \qquad e^{-|x|} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{4}\frac{P(H_0)}{1-P(H_0)}$$

Taking the natural logarithm of both sides of the last expression yields

$$|x| \underset{H_0}{\overset{H_1}{\lesssim}} \ln\frac{4[1-P(H_0)]}{P(H_0)} = \delta \tag{*8.34*}$$

For $P(H_0) > 0.8$, $\delta$ becomes negative, and we always decide $H_0$. For $P(H_0) \le 0.8$, the decision regions are

$$R_0 = \{x: |x| > \delta\} \qquad R_1 = \{x: |x| < \delta\}$$

Then, by setting $C_{00} = C_{11} = 0$, $C_{01} = 2$, and $C_{10} = 1$ in Eq. (8.19), the minimum Bayes' risk $\bar{C}^*$ can be expressed as a function of $P(H_0)$ as

$$\begin{aligned}\bar{C}^*[P(H_0)] &= P(H_0)\int_{-\delta}^{\delta} \tfrac{1}{2}e^{-|x|}\,dx + 2[1 - P(H_0)]\left[\int_{-\infty}^{-\delta} e^{2x}\,dx + \int_{\delta}^{\infty} e^{-2x}\,dx\right] \\ &= P(H_0)\int_{0}^{\delta} e^{-x}\,dx + 4[1 - P(H_0)]\int_{\delta}^{\infty} e^{-2x}\,dx \\ &= P(H_0)(1 - e^{-\delta}) + 2[1 - P(H_0)]e^{-2\delta} \qquad (8.35)\end{aligned}$$

From the definition of $\delta$ [Eq. (8.34)], we have

$$e^{\delta} = \frac{4[1 - P(H_0)]}{P(H_0)}$$

Thus $$e^{-\delta} = \frac{P(H_0)}{4[1 - P(H_0)]} \quad \text{and} \quad e^{-2\delta} = \frac{P^2(H_0)}{16[1 - P(H_0)]^2}$$

Substituting these values into Eq. (8.35), we obtain

$$\bar{C}^*[P(H_0)] = \frac{8P(H_0) - 9P^2(H_0)}{8[1 - P(H_0)]}$$

Now the value of $P(H_0)$ which maximizes $\bar{C}^*$ can be obtained by setting $d\bar{C}^*[P(H_0)]/dP(H_0)$ equal to zero and solving for $P(H_0)$. The result yields $P(H_0) = \frac{2}{3}$. Substituting this value into Eq. (8.34), we obtain the following minimax test:

$$|x| \underset{H_0}{\overset{H_1}{\lesssim}} \ln \frac{4(1 - \frac{2}{3})}{\frac{2}{3}} = \ln 2 = 0.69$$

**8.13.** Suppose that we have $n$ observations $X_i$, $i = 1, \ldots, n$, of radar signals, and $X_i$ are normal iid r.v.'s under each hypothesis. Under $H_0$, $X_i$ have mean $\mu_0$ and variance $\sigma^2$, while under $H_1$, $X_i$ have mean $\mu_1$ and variance $\sigma^2$, and $\mu_1 > \mu_0$. Determine the maximum likelihood test.

By Eq. (2.52) for each $X_i$, we have

$$f(x_i|H_0) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x_i - \mu_0)^2\right]$$

$$f(x_i|H_1) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}(x_i - \mu_1)^2\right]$$

Since the $X_i$ are independent, we have

$$f(\mathbf{x}|H_0) = \prod_{i=1}^{n} f(x_i|H_0) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_0)^2\right]$$

$$f(\mathbf{x}|H_1) = \prod_{i=1}^{n} f(x_i|H_1) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu_1)^2\right]$$

With $\mu_1 - \mu_0 > 0$, the likelihood ratio is then given by

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x}|H_1)}{f(\mathbf{x}|H_0)} = \exp\left\{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} 2(\mu_1 - \mu_0)x_i - n(\mu_1^2 - \mu_0^2)\right]\right\}$$

Hence, the maximum likelihood test is given by

$$\exp\left\{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n} 2(\mu_1 - \mu_0)x_i - n(\mu_1^2 - \mu_0^2)\right]\right\} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Taking the natural logarithm of both sides of the above expression yields

$$\frac{\mu_1 - \mu_0}{\sigma^2} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{n(\mu_1^2 - \mu_0^2)}{2\sigma^2}$$

or

$$\frac{1}{n} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\mu_1 + \mu_0}{2} \tag{8.36}$$

Equation (8.36) indicates that the statistic

$$s(X, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i = \bar{X}$$

provides enough information about the observations to enable us to make a decision. Thus, it is called the *sufficient statistic* for the maximum likelihood test.

**8.14.** Consider the same observations $X_i$, $i = 1, \ldots, n$, of radar signals as in Prob. 8.13, but now, under $H_0$, $X_i$ have zero mean and variance $\sigma_0^2$, while under $H_1$, $X_i$ have zero mean and variance $\sigma_1^2$, and $\sigma_1^2 > \sigma_0^2$. Determine the maximum likelihood test.

In a similar manner as in Prob. 8.13, we obtain

$$f(\mathbf{x} \mid H_0) = \frac{1}{(2\pi\sigma_0^2)^{n/2}} \exp\left(-\frac{1}{2\sigma_0^2} \sum_{i=1}^{n} x_i^2\right)$$

$$f(\mathbf{x} \mid H_1) = \frac{1}{(2\pi\sigma_1^2)^{n/2}} \exp\left(-\frac{1}{2\sigma_1^2} \sum_{i=1}^{n} x_i^2\right)$$

With $\sigma_1^2 - \sigma_0^2 > 0$, the likelihood ratio is

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left[\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \sum_{i=1}^{n} x_i^2\right]$$

and the maximum likelihood test is

$$\left(\frac{\sigma_0}{\sigma_1}\right)^n \exp\left[\left(\frac{\sigma_1^2 - \sigma_0^2}{2\sigma_0^2\sigma_1^2}\right) \sum_{i=1}^{n} x_i^2\right] \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Taking the natural logarithm of both sides of the above expression yields

$$\sum_{i=1}^{n} x_i^2 \underset{H_0}{\overset{H_1}{\gtrless}} n\left[\ln\left(\frac{\sigma_1}{\sigma_0}\right)\right]\left(\frac{2\sigma_0^2\sigma_1^2}{\sigma_1^2 - \sigma_0^2}\right) \tag{8.37}$$

Note that in this case,

$$s(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i^2$$

is the sufficient statistic for the maximum likelihood test.

**8.15.** In the binary communication system of Prob. 8.6, suppose that we have $n$ independent observations $X_i = X(t_i)$, $i = 1, \ldots, n$, where $0 < t_1 < \cdots < t_n \leq T$.

(*a*) Determine the maximum likelihood test.

(*b*) Find $P_I$ and $P_{II}$ for $n = 5$ and $n = 10$.

(*a*) Setting $\mu_0 = 0$ and $\mu_1 = 1$ in Eq. (8.36), the maximum likelihood test is

$$\frac{1}{n} \sum_{i=1}^{n} x_i \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}$$

(*b*) Let $$Y = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Then by Eqs. (*4.108*) and (*4.112*), and the result of Prob. 5.60, we see that $Y$ is a normal r.v. with zero mean and variance $1/n$ under $H_0$, and is a normal r.v. with mean 1 and variance $1/n$ under $H_1$. Thus

$$P_I = P(D_1 \mid H_0) = \int_{R_1} f_Y(y \mid H_0)\, dy = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-(n/2)y^2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{n}/2}^{\infty} e^{-z^2/2}\, dz = 1 - \Phi(\sqrt{n}/2)$$

$$P_{II} = P(D_0 \mid H_1) = \int_{R_0} f_Y(y \mid H_1)\, dy = \frac{\sqrt{n}}{\sqrt{2\pi}} \int_{1/2}^{\infty} e^{-(n/2)(y-1)^2}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\sqrt{n}/2} e^{-z^2/2}\, dz = \Phi(-\sqrt{n}/2) = 1 - \Phi(\sqrt{n}/2)$$

Note that $P_I = P_{II}$. Using Table A (Appendix A), we have

$$P_I = P_{II} = 1 - \Phi(1.118) = 0.1318 \qquad \text{for } n = 5$$
$$P_I = P_{II} = 1 - \Phi(1.581) = 0.057 \qquad \text{for } n = 10$$

**8.16.** In the binary communication system of Prob. 8.6, suppose that $s_0(t)$ and $s_1(t)$ are arbitrary signals and that $n$ observations of the received signal $x(t)$ are made. Let $n$ samples of $s_0(t)$ and $s_1(t)$ be represented, respectively, by

$$\mathbf{s}_0 = [s_{01}, s_{02}, \ldots, s_{0n}]^T \qquad \text{and} \qquad \mathbf{s}_1 = [s_{11}, s_{12}, \ldots, s_{1n}]^T$$

where $T$ denotes "transpose of." Determine the MAP test.

For each $X_i$, we can write

$$f(x_i \mid H_0) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(x_i - s_{0i})^2\right]$$
$$f(x_i \mid H_1) = \frac{1}{\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}(x_i - s_{1i})^2\right]$$

Since the noise components are independent, we have

$$f(\mathbf{x} \mid H_j) = \prod_{i=1}^{n} f(x_i \mid H_j) \qquad j = 0, 1$$

and the likelihood ratio is given by

$$\Lambda(\mathbf{x}) = \frac{f(\mathbf{x} \mid H_1)}{f(\mathbf{x} \mid H_0)} = \frac{\prod_{i=1}^{n} \exp\left[-\tfrac{1}{2}(x_i - s_{1i})^2\right]}{\prod_{i=1}^{n} \exp\left[-\tfrac{1}{2}(x_i - s_{0i})^2\right]}$$
$$= \exp\left[\sum_{i=1}^{n} (s_{1i} - s_{0i})x_i - \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2)\right]$$

Thus, by Eq. (*8.15*), the MAP test is given by

$$\exp\left[\sum_{i=1}^{n} (s_{1i} - s_{0i})x_i - \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{P(H_0)}{P(H_1)}$$

Taking the natural logarithm of both sides of the above expression yields

$$\sum_{i=1}^{n} (s_{1i} - s_{0i})x_i \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left[\frac{P(H_0)}{P(H_1)}\right] + \tfrac{1}{2}(s_{1i}^2 - s_{0i}^2) \tag{8.38}$$

# Supplementary Problems

**8.17.** Let $(X_1, \ldots, X_n)$ be a random sample of a Bernoulli r.v. $X$ with pmf

$$f(x; p) = p^x(1 - p)^{1-x} \qquad x = 0, 1$$

where it is known that $0 < p \leq \frac{1}{2}$. Let

$$H_0: \quad p = \tfrac{1}{2}$$
$$H_1: \quad p = p_1 \ (< \tfrac{1}{2})$$

and $n = 20$. As a decision test, we use the rule to reject $H_0$ if $\sum_{i=1}^{n} x_i \leq 6$.

(*a*) Find the power function $g(p)$ of the test.

(*b*) Find the probability of a Type I error $\alpha$.

(*c*) Find the probability of a Type II error $\beta$ (i) when $p_1 = \frac{1}{4}$ and (ii) when $p_1 = \frac{1}{10}$.

*Ans.* (*a*) $g(p) = \sum_{k=0}^{6} \binom{20}{k} p^k(1 - p)^{20-k} \qquad 0 < p \leq \frac{1}{2}$

(*b*) $\alpha = 0.0577$; (*c*) (i) $\beta = 0.2142$, (ii) $\beta = 0.0024$

**8.18.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. $X$ with mean $\mu$ and variance 36. Let

$$H_0: \quad \mu = 50$$
$$H_1: \quad \mu = 55$$

As a decision test, we use the rule to accept $H_0$ if $\bar{x} < 53$, where $\bar{x}$ is the value of the sample mean.

(*a*) Find the expression for the critical region $R_1$.

(*b*) Find $\alpha$ and $\beta$ for $n = 16$.

*Ans.* (*a*) $R_1 = \{(x_1, \ldots, x_N); \bar{x} \geq 53\}$ where $\bar{x} = \dfrac{1}{n} \sum_{n=1}^{n} x_i$

(*b*) $\alpha = 0.0228$, $\beta = 0.0913$

**8.19.** Let $(X_1, \ldots, X_n)$ be a random sample of a normal r.v. $X$ with mean $\mu$ and variance 100. Let

$$H_0: \quad \mu = 50$$
$$H_1: \quad \mu = 55$$

As a decision test, we use the rule that we reject $H_0$ if $\bar{x} \geq c$. Find the value of $c$ and sample size $n$ such that $\alpha = 0.025$ and $\beta = 0.05$.

*Ans.* $c = 52.718$, $n = 52$

**8.20.** Let $X$ be a normal r.v. with zero mean and variance $\sigma^2$. Let

$$H_0: \quad \sigma^2 = 1$$
$$H_1: \quad \sigma^2 = 4$$

Determine the maximum likelihood test.

*Ans.* $|x| \underset{H_0}{\overset{H_1}{\gtrless}} 1.36$

**8.21.** Consider the binary decision problem of Prob. 8.20. Let $P(H_0) = \frac{2}{3}$ and $P(H_1) = \frac{1}{3}$. Determine the MAP test.

*Ans.* $|x| \underset{H_0}{\overset{H_1}{\gtrless}} 1.923$

**8.22.** Consider the binary communication system of Prob. 8.6.

(*a*) Construct a Neyman-Pearson test for the case where $\alpha = 0.1$.

(*b*) Find $\beta$.

*Ans.* (*a*) $|x| \underset{H_0}{\overset{H_1}{\gtrless}} 1.282$; (*b*) $\beta = 0.6111$

**8.23.** Consider the binary decision problem of Prob. 8.11. Determine the Bayes' test if $P(H_0) = 0.25$ and the Bayes' costs are

$$C_{00} = C_{11} = 0 \qquad C_{01} = 1 \qquad C_{10} = 2$$

*Ans.* $|x| \underset{H_0}{\overset{H_1}{\lessgtr}} 1.10$

# Chapter 9

# Queueing Theory

## 9.1 INTRODUCTION

Queueing theory deals with the study of queues (waiting lines). Queues abound in practical situations. The earliest use of queueing theory was in the design of a telephone system. Applications of queueing theory are found in fields as seemingly diverse as traffic control, hospital management, and time-shared computer system design. In this chapter, we present an elementary queueing theory.

## 9.2 QUEUEING SYSTEMS

### A. Description:

A simple queueing system is shown in Fig. 9-1. Customers arrive randomly at an average rate of $\lambda_a$. Upon arrival, they are served without delay if there are available servers; otherwise, they are made to wait in the queue until it is their turn to be served. Once served, they are assumed to leave the system. We will be interested in determining such quantities as the average number of customers in the system, the average time a customer spends in the system, the average time spent waiting in the queue, etc.

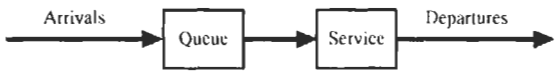



**Fig. 9-1** A simple queueing system.

The description of any queueing system requires the specification of three parts:

1. The arrival process
2. The service mechanism, such as the number of servers and service-time distribution
3. The queue discipline (for example, first-come, first-served)

### B. Classification:

The notation $A/B/s/K$ is used to classify a queueing system, where $A$ specifies the type of arrival process, $B$ denotes the service-time distribution, $s$ specifies the number of servers, and $K$ denotes the capacity of the system, that is, the maximum number of customers that can be accommodated. If $K$ is not specified, it is assumed that the capacity of the system is unlimited. For example, an M/M/2 queueing system (M stands for Markov) is one with Poisson arrivals (or exponential interarrival time distribution), exponential service-time distribution, and 2 servers. An M/G/1 queueing system has Poisson arrivals, general service-time distribution, and a single server. A special case is the M/D/1 queueing system, where D stands for constant (deterministic) service time. Examples of queueing systems with limited capacity are telephone systems with limited trunks, hospital emergency rooms with limited beds, and airline terminals with limited space in which to park aircraft for loading and unloading. In each case, customers who arrive when the system is saturated are denied entrance and are lost.

### C. Useful Formulas

Some basic quantities of queueing systems are

$L$: the average number of customers in the system

$L_q$: the average number of customers waiting in the queue

$L_s$: the average number of customers in service

$W$: the average amount of time that a customer spends in the system

$W_q$: the average amount of time that a customer spends waiting in the queue

$W_s$: the average amount of time that a customer spends in service

Many useful relationships between the above and other quantities of interest can be obtained by using the following basic cost identity:

Assume that entering customers are required to pay an entrance fee (according to some rule) to the system. Then we have

$$\text{Average rate at which the system earns} = \lambda_a \times \text{average amount an entering customer pays} \tag{9.1}$$

where $\lambda_a$ is the average arrival rate of entering customers

$$\lambda_a = \lim_{t \to \infty} \frac{X(t)}{t}$$

and $X(t)$ denotes the number of customer arrivals by time $t$.

If we assume that each customer pays \$1 per unit time while in the system, Eq. (9.1) yields

$$L = \lambda_a W \tag{9.2}$$

Equation (9.2) is sometimes known as *Little's formula.*

Similarly, if we assume that each customer pays \$1 per unit time while in the queue, Eq. (9.1) yields

$$L_q = \lambda_a W_q \tag{9.3}$$

If we assume that each customer pays \$1 per unit time while in service, Eq. (9.1) yields

$$L_s = \lambda_a W_s \tag{9.4}$$

Note that Eqs. (9.2) to (9.4) are valid for almost all queueing systems, regardless of the arrival process, the number of servers, or queueing discipline.

## 9.3 BIRTH-DEATH PROCESS

We say that the queueing system is in state $S_n$ if there are $n$ customers in the system, including those being served. Let $N(t)$ be the Markov process that takes on the value $n$ when the queueing system is in state $S_n$ with the following assumptions:

1. If the system is in state $S_n$, it can make transitions only to $S_{n-1}$ or $S_{n+1}$, $n \geq 1$; that is, either a customer completes service and leaves the system or, while the present customer is still being serviced, another customer arrives at the system; from $S_0$, the next state can only be $S_1$.
2. If the system is in state $S_n$ at time $t$, the probability of a transition to $S_{n+1}$ in the time interval $(t, t + \Delta t)$ is $a_n \Delta t$. We refer to $a_n$ as the *arrival* parameter (or the *birth* parameter).
3. If the system is in state $S_n$ at time $t$, the probability of a transition to $S_{n-1}$ in the time interval $(t, t + \Delta t)$ is $d_n \Delta t$. We refer to $d_n$ as the *departure* parameter (or the *death* parameter).

The process $N(t)$ is sometimes referred to as the *birth-death* process.

Let $p_n(t)$ be the probability that the queueing system is in state $S_n$ at time $t$; that is,

$$p_n(t) = P\{N(t) = n\} \tag{9.5}$$

Then we have the following fundamental recursive equations for $N(t)$ (Prob. 9.2):

$$\begin{aligned} p_n'(t) &= -(a_n + d_n)p_n(t) + a_{n-1}p_{n-1}(t) + d_{n+1}p_{n+1}(t) \qquad n \geq 1 \\ p_0'(t) &= -(a_0 + d_0)p_0(t) + d_1 p_1(t) \end{aligned} \tag{9.6}$$

Assume that in the steady state we have

$$\lim_{t\to\infty} p_n(t) = p_n \tag{9.7}$$

and setting $p_0'(t)$ and $p_n'(t) = 0$ in Eqs. (9.6), we obtain the following steady-state recursive equation:

$$(a_n + d_n)p_n = a_{n-1}p_{n-1} + d_{n+1}p_{n+1} \qquad n \geq 1 \tag{9.8}$$

and for the special case with $d_0 = 0$,

$$a_0 p_0 = d_1 p_1 \tag{9.9}$$

Equations (9.8) and (9.9) are also known as the steady-state *equilibrium equations*. The state transition diagram for the birth-death process is shown in Fig. 9-2.

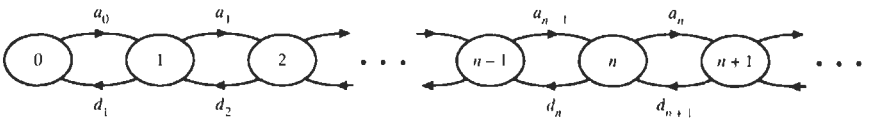



**Fig. 9-2** State transition diagram for the birth-death process.

Solving Eqs. (9.8) and (9.9) in terms of $p_0$, we obtain

$$\begin{aligned} p_1 &= \frac{a_0}{d_1} p_0 \\ p_2 &= \frac{a_0 a_1}{d_1 d_2} p_0 \\ &\vdots \\ p_n &= \frac{a_0 a_1 \cdots a_{n-1}}{d_1 d_2 \cdots d_n} p_0 \end{aligned} \tag{9.10}$$

where $p_0$ can be determined from the fact that

$$\sum_{n=0}^{\infty} p_n = \left(1 + \frac{a_0}{d_1} + \frac{a_0 a_1}{d_1 d_2} + \cdots\right) p_0 = 1 \tag{9.11}$$

provided that the summation in parentheses converges to a finite value.

## 9.4 THE M/M/1 QUEUEING SYSTEM

In the M/M/1 queueing system, the arrival process is the Poisson process with rate $\lambda$ (the mean arrival rate) and the service time is exponentially distributed with parameter $\mu$ (the mean service rate). Then the process $N(t)$ describing the state of the M/M/1 queueing system at time $t$ is a birth-death process with the following state independent parameters:

$$a_n = \lambda \qquad n \geq 0 \qquad\qquad d_n = \mu \qquad n \geq 1 \tag{9.12}$$

Then from Eqs. (9.10) and (9.11), we obtain (Prob. 9.3)

$$p_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho \tag{9.13}$$

$$p_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n = (1 - \rho)\rho^n \tag{9.14}$$

where $\rho = \lambda/\mu < 1$, which implies that the server, on the average, must process the customers faster than their average arrival rate; otherwise the queue length (the number of customers waiting in the queue) tends to infinity. The ratio $\rho = \lambda/\mu$ is sometimes referred to as the *traffic intensity* of the

system. The traffic intensity of the system is defined as

$$\text{Traffic intensity} = \frac{\text{mean service time}}{\text{mean interarrival time}} = \frac{\text{mean arrival rate}}{\text{mean service rate}}$$

The average number of customers in the system is given by (Prob. 9.4)

$$L = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu - \lambda} \tag{9.15}$$

Then, setting $\lambda_a = \lambda$ in Eqs. (9.2) to (9.4), we obtain (Prob. 9.5)

$$W = \frac{1}{\mu - \lambda} = \frac{1}{\mu(1-\rho)} \tag{9.16}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu(1-\rho)} \tag{9.17}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1-\rho} \tag{9.18}$$

## 9.5 THE M/M/$s$ QUEUEING SYSTEM

In the M/M/$s$ queueing system, the arrival process is the Poisson process with rate $\lambda$ and each of the $s$ servers has an exponential service time with parameter $\mu$. In this case, the process $N(t)$ describing the state of the M/M/$s$ queueing system at time $t$ is a birth-death process with the following parameters:

$$a_n = \lambda \qquad n \geq 0 \qquad\qquad d_n = \begin{cases} n\mu & 0 < n < s \\ s\mu & n \geq s \end{cases} \tag{9.19}$$

Note that the departure parameter $d_n$ is state dependent. Then, from Eqs. (9.10) and (9.11), we obtain (Prob. 9.10)

$$p_0 = \left[ \sum_{n=0}^{s-1} \frac{(s\rho)}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right]^{-1} \tag{9.20}$$

$$p_n = \begin{cases} \dfrac{(s\rho)^n}{n!} p_0 & n < s \\ \dfrac{\rho^n s^s}{s!} p_0 & n \geq s \end{cases} \tag{9.21}$$

where $\rho = \lambda/(s\mu) < 1$. Note that the ratio $\rho = \lambda/(s\mu)$ is the traffic intensity of the M/M/$s$ queueing system. The average number of customers in the system and the average number of customers in the queue are given, respectively, by (Prob. 9.12)

$$L = \frac{\lambda}{\mu} + \frac{\rho(s\rho)^s}{s!(1-\rho)^2} p_0 \tag{9.22}$$

$$L_q = \frac{\rho(s\rho)^s}{s!(1-\rho)^2} p_0 = L - \frac{\lambda}{\mu} \tag{9.23}$$

By Eqs. (9.2) and (9.3), the quantities $W$ and $W_q$ are given by

$$W = \frac{L}{\lambda} \tag{9.24}$$

$$W_q = \frac{L_q}{\lambda} = W - \frac{1}{\mu} \tag{9.25}$$

## 9.6 THE M/M/1/$K$ QUEUEING SYSTEM

In the M/M/1/$K$ queueing system, the capacity of the system is limited to $K$ customers. When the system reaches its capacity, the effect is to reduce the arrival rate to zero until such time as a customer is served to again make queue space available. Thus, the M/M/1/$K$ queueing system can be modeled as a birth-death process with the following parameters:

$$a_n = \begin{cases} \lambda & 0 \le n < K \\ 0 & n \ge K \end{cases} \qquad d_n = \mu \qquad n \ge 1 \tag{9.26}$$

Then, from Eqs. (*9.10*) and (*9.11*), we obtain (Prob. 9.14)

$$p_0 = \frac{1 - (\lambda/\mu)}{1 - (\lambda/\mu)^{K+1}} = \frac{1 - \rho}{1 - \rho^{K+1}} \qquad \rho \ne 1 \tag{9.27}$$

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \frac{(1 - \rho)\rho^n}{1 - \rho^{K+1}} \qquad n = 1, \ldots, K \tag{9.28}$$

where $\rho = \lambda/\mu$. It is important to note that it is no longer necessary that the traffic intensity $\rho = \lambda/\mu$ be less than 1. Customers are denied service when the system is in state $K$. Since the fraction of arrivals that actually enter the system is $1 - p_K$, the effective arrival rate is given by

$$\lambda_e = \lambda(1 - p_K) \tag{9.29}$$

The average number of customers in the system is given by (Prob. 9.15)

$$L = \rho \frac{1 - (K + 1)\rho^K + K\rho^{K+1}}{(1 - \rho)(1 - \rho^{K+1})} \qquad \rho = \frac{\lambda}{\mu} \tag{9.30}$$

Then, setting $\lambda_a = \lambda_e$ in Eqs. (*9.2*) to (*9.4*), we obtain

$$W = \frac{L}{\lambda_e} = \frac{L}{\lambda(1 - p_K)} \tag{9.31}$$

$$W_q = W - \frac{1}{\mu} \tag{9.32}$$

$$L_q = \lambda_e W_q = \lambda(1 - p_K)W_q \tag{9.33}$$

## 9.7 THE M/M/$s$/$K$ QUEUEING SYSTEM

Similarly, the M/M/$s$/$K$ queueing system can be modeled as a birth-death process with the following parameters:

$$a_n = \begin{cases} \lambda & 0 \le n < K \\ 0 & n \ge K \end{cases} \qquad d_n = \begin{cases} n\mu & 0 < n < s \\ s\mu & n \ge s \end{cases} \tag{9.34}$$

Then, from Eqs. (*9.10*) and (*9.11*), we obtain (Prob. 9.17)

$$p_0 = \left[\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!}\left(\frac{1 - \rho^{K-s+1}}{1 - \rho}\right)\right]^{-1} \tag{9.35}$$

$$p_n = \begin{cases} \dfrac{(s\rho)^n}{n!} p_0 & n < s \\ \dfrac{\rho^n s^s}{s!} p_0 & s \le n \le K \end{cases} \tag{9.36}$$

where $\rho = \lambda/(s\mu)$. Note that the expression for $p_n$ is identical in form to that for the M/M/$s$ system, Eq. (*9.21*). They differ only in the $p_0$ term. Again, it is not necessary that $\rho = \lambda/(s\mu)$ be less than 1. The

average number of customers in the queue is given by (Prob. 9.18)

$$L_q = p_0 \frac{\rho(\rho s)^s}{s!(1-\rho)^2} \{1 - [1 + (1-\rho)(K-s)]\rho^{K-s}\} \tag{9.37}$$

The average number of customers in the system is

$$L = L_q + \frac{\lambda_e}{\mu} = L_q + \frac{\lambda}{\mu}(1 - p_K) \tag{9.38}$$

The quantities $W$ and $W_q$ are given by

$$W = \frac{L}{\lambda_e} = L_q + \frac{1}{\mu} \tag{9.39}$$

$$W_q = \frac{L_q}{\lambda_e} = \frac{L_q}{\lambda(1 - p_K)} \tag{9.40}$$

## Solved Problems

**9.1.** Deduce the basic cost identity (9.1).

Let $T$ be a fixed large number. The amount of money earned by the system by time $T$ can be computed by multiplying the average rate at which the system earns by the length of time $T$. On the other hand, it can also be computed by multiplying the average amount paid by an entering customer by the average number of customers entering by time $T$, which is equal to $\lambda_a T$, where $\lambda_a$ is the average arrival rate of entering customers. Thus, we have

Average rate at which the system earns $\times$ $T$ = average amount paid by an entering customer $\times$ $(\lambda_a T)$

Dividing both sides by $T$ (and letting $T \to \infty$), we obtain Eq. (9.1).

**9.2.** Derive Eq. (9.6).

From properties 1 to 3 of the birth-death process $N(t)$, we see that at time $t + \Delta t$, the system can be in state $S_n$ in three ways:

1. By being in state $S_n$ at time $t$ and no transition occurring in the time interval $(t, t + \Delta t)$. This happens with probability $(1 - a_n \Delta t)(1 - d_n \Delta t) \approx 1 - (a_n + d_n)\Delta t$ [by neglecting the second-order effect $a_n d_n (\Delta t)^2$].
2. By being in state $S_{n-1}$ at time $t$ and a transition to $S_n$ occurring in the time interval $(t, t + \Delta t)$. This happens with probability $a_{n-1}\Delta t$.
3. By being in state $S_{n+1}$ at time $t$ and a transition to $S_n$ occurring in the time interval $(t, t + \Delta t)$. This happens with probability $d_{n+1}\Delta t$.

Let $$p_i(t) = P[N(t) = i]$$

Then, using the Markov property of $N(t)$, we obtain

$$p_n(t + \Delta t) = [1 - (a_n + d_n)\Delta t]p_n(t) + a_{n-1}\Delta t\, p_{n-1}(t) + d_{n+1}\Delta t\, p_{n+1}(t) \qquad n \geq 1$$
$$p_0(t + \Delta t) = [1 - (a_0 + d_0)\Delta t]p_0(t) + d_1 \Delta t\, p_1(t)$$

Rearranging the above relations

$$\frac{p_n(t+\Delta t)-p_n(t)}{\Delta t} = -(a_n+d_n)p_n(t) + a_{n-1}p_{n-1}(t) + d_{n+1}p_{n+1}(t) \qquad n \geq 1$$

$$\frac{p_0(t+\Delta t)-p_0(t)}{\Delta t} = -(a_0+d_0)p_0(t) + d_1p_1(t)$$

Letting $\Delta t \to 0$, we obtain

$$p_n'(t) = -(a_n+d_n)p_n(t) + a_{n-1}p_{n-1}(t) + d_{n+1}p_{n+1}(t) \qquad n \geq 1$$

$$p_0'(t) = -(a_0+d_0)p_0(t) + d_1p_1(t)$$

**9.3.** Derive Eqs. (*9.13*) and (*9.14*).

Setting $a_n = \lambda$, $d_0 = 0$, and $d_n = \mu$ in Eq. (*9.10*), we get

$$p_1 = \frac{\lambda}{\mu}p_0 = \rho p_0$$

$$p_2 = \left(\frac{\lambda}{\mu}\right)^2 p_0 = \rho^2 p_0$$

$$\vdots$$

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0$$

where $p_0$ is determined by equating

$$\sum_{n=0}^{\infty} p_n = p_0 \sum_{n=0}^{\infty} \rho^n = p_0 \frac{1}{1-\rho} = 1 \qquad |\rho| < 1$$

from which we obtain

$$p_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}$$

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = (1-\rho)\rho^n = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

**9.4.** Derive Eq. (*9.15*).

Since $p_n$ is the steady-state probability that the system contains exactly $n$ customers, using Eq. (*9.14*), the average number of customers in the M/M/1 queueing system is given by

$$L = \sum_{n=0}^{\infty} np_n = \sum_{n=0}^{\infty} n(1-\rho)\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n \tag{9.41}$$

where $\rho = \lambda/\mu < 1$. Using the algebraic identity

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2} \qquad |x| < 1 \tag{9.42}$$

we obtain

$$L = \frac{\rho}{1-\rho} = \frac{\lambda/\mu}{1-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

**9.5.** Derive Eqs. (*9.16*) to (*9.18*).

Since $\lambda_a = \lambda$, by Eqs. (*9.2*) and (*9.15*), we get

$$W = \frac{L}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

which is Eq. (9.16). Next, by definition,

$$W_q = W - W_s \tag{9.43}$$

where $W_s = 1/\mu$, that is, the average service time. Thus,

$$W_q = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu(1 - \rho)}$$

which is Eq. (9.17). Finally, by Eq. (9.3),

$$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{\rho^2}{1 - \rho}$$

which is Eq. (9.18).

**9.6.** Let $W_a$ denote the amount of time an arbitrary customer spends in the M/M/1 queueing system. Find the distribution of $W_a$.

We have

$$P\{W_a \le a\} = \sum_{n=0}^{\infty} P\{W_a \le a \mid n \text{ in the system when the customer arrives}\} \times P\{n \text{ in the system when the customer arrives}\} \tag{9.44}$$

where $n$ is the number of customers in the system. Now consider the amount of time $W_a$ that this customer will spend in the system when there are already $n$ customers when he or she arrives. When $n = 0$, then $W_a = W_{s(a)}$, that is, the service time. When $n \ge 1$, there will be one customer in service and $n - 1$ customers waiting in line ahead of this customer's arrival. The customer in service might have been in service for some time, but because of the memoryless property of the exponential distribution of the service time, it follows that (see Prob. 2.57) the arriving customer would have to wait an exponential amount of time with parameter $\mu$ for this customer to complete service. In addition, the customer also would have to wait an exponential amount of time for each of the other $n - 1$ customers in line. Thus, adding his or her own service time, the amount of time $W_a$ that the customer will spend in the system when there are already $n$ customers when he or she arrives is the sum of $n + 1$ iid exponential r.v.'s with parameter $\mu$. Then by Prob. 4.33, we see that this r.v. is a gamma r.v. with parameters $(n + 1, \mu)$. Thus, by Eq. (2.83),

$$P\{W_a \le a \mid n \text{ in the system when customer arrives}\} = \int_0^a \mu e^{-\mu t} \frac{(\mu t)^n}{n!}\, dt$$

From Eq. (9.14),

$$P\{n \text{ in the system when customer arrives}\} = p_n = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n$$

Hence, by Eq. (9.44),

$$\begin{aligned} F_{W_a} = P\{W_a \le a\} &= \sum_{n=0}^{\infty} \left[\int_0^a \mu e^{-\mu t} \frac{(\mu t)^n}{n!} \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^n dt\right] \\ &= \int_0^a (\mu - \lambda) e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\, dt \\ &= \int_0^a (\mu - \lambda) e^{-(\mu - \lambda)t}\, dt = 1 - e^{-(\mu - \lambda)a} \end{aligned} \tag{9.45}$$

Thus, by Eq. (2.79), $W_a$ is an exponential r.v. with parameter $\mu - \lambda$. Note that from Eq. (2.99), $E(W_a) = 1/(\mu - \lambda)$, which agrees with Eq. (9.16), since $W = E(W_a)$.

**9.7.** Customers arrive at a watch repair shop according to a Poisson process at a rate of one per every 10 minutes, and the service time is an exponential r.v. with mean 8 minutes.

(*a*) Find the average number of customers $L$, the average time a customer spends in the shop $W$, and the average time a customer spends in waiting for service $W_q$.

(*b*) Suppose that the arrival rate of the customers increases 10 percent. Find the corresponding changes in $L$, $W$, and $W_q$.

(*a*) The watch repair shop service can be modeled as an M/M/1 queueing system with $\lambda = \frac{1}{10}$, $\mu = \frac{1}{8}$. Thus, from Eqs. (*9.15*), (*9.16*), and (*9.43*), we have

$$L = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{10}}{\frac{1}{8} - \frac{1}{10}} = 4$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{8} - \frac{1}{10}} = 40 \text{ minutes}$$

$$W_q = W - W_s = 40 - 8 = 32 \text{ minutes}$$

(*b*) Now $\lambda = \frac{1}{9}$, $\mu = \frac{1}{8}$. Then

$$L = \frac{\lambda}{\mu - \lambda} = \frac{\frac{1}{9}}{\frac{1}{8} - \frac{1}{9}} = 8$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{8} - \frac{1}{9}} = 72 \text{ minutes}$$

$$W_q = W - W_s = 72 - 8 = 64 \text{ minutes}$$

It can be seen that an increase of 10 percent in the customer arrival rate doubles the average number of customers in the system. The average time a customer spends in queue is also doubled.

**9.8.** A drive-in banking service is modeled as an M/M/1 queueing system with customer arrival rate of 2 per minute. It is desired to have fewer than 5 customers line up 99 percent of the time. How fast should the service rate be?

From Eq. (*9.14*),

$$P\{5 \text{ or more customers in the system}\} = \sum_{n=5}^{\infty} p_n = \sum_{n=5}^{\infty} (1 - \rho)\rho^n = \rho^5 \qquad \rho = \frac{\lambda}{\mu}$$

In order to have fewer than 5 customers line up 99 percent of the time, we require that this probability be less than 0.01. Thus,

$$\rho^5 = \left(\frac{\lambda}{\mu}\right)^5 \le 0.01$$

from which we obtain

$$\mu^5 \ge \frac{\lambda^5}{0.01} = \frac{2^5}{0.01} = 3200 \qquad \text{or} \qquad \mu \ge 5.024$$

Thus, to meet the requirements, the average service rate must be at least 5.024 customers per minute.

**9.9.** People arrive at a telephone booth according to a Poisson process at an average rate of 12 per hour, and the average time for each call is an exponential r.v. with mean 2 minutes.

(*a*) What is the probability that an arriving customer will find the telephone booth occupied?

(*b*) It is the policy of the telephone company to install additional booths if customers wait an average of 3 or more minutes for the phone. Find the average arrival rate needed to justify a second booth.

(*a*) The telephone service can be modeled as an M/M/1 queueing system with $\lambda = \frac{1}{5}$, $\mu = \frac{1}{2}$, and $\rho = \lambda/\mu = \frac{2}{5}$. The probability that an arriving customer will find the telephone occupied is $P(L > 0)$, where $L$ is the average number of customers in the system. Thus, from Eq. (*9.13*),

$$P(L > 0) = 1 - p_0 = 1 - (1 - \rho) = \rho = \tfrac{2}{5} = 0.4$$

(*b*) From Eq. (*9.17*),

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\lambda}{0.5(0.5 - \lambda)} \geq 3$$

from which we obtain $\lambda \geq 0.3$ per minute. Thus, the required average arrival rate to justify the second booth is 18 per hour.

**9.10.** Derive Eqs. (*9.20*) and (*9.21*).

From Eqs. (*9.19*) and (*9.10*), we have

$$p_n = p_0 \prod_{k=0}^{n-1} \frac{\lambda}{(k+1)\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \qquad n < s \tag{9.46}$$

$$p_n = p_0 \prod_{k=0}^{s-1} \frac{\lambda}{(k+1)\mu} \prod_{k=s}^{n-1} \frac{\lambda}{s\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{s!\, s^{n-s}} \qquad n \geq s \tag{9.47}$$

Let $\rho = \lambda/(s\mu)$. Then Eqs. (*9.46*) and (*9.47*) can be rewritten as

$$p_n = \begin{cases} \dfrac{(s\rho)^n}{n!} p_0 & n < s \\ \dfrac{\rho^n s^s}{s!} p_0 & n \geq s \end{cases}$$

which is Eq. (*9.21*). From Eq. (*9.11*), $p_0$ is obtained by equating

$$\sum_{n=0}^{\infty} p_n = p_0 \left[ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \sum_{n=s}^{\infty} \frac{\rho^n s^s}{s!} \right] = 1$$

Using the summation formula

$$\sum_{n=k}^{\infty} x^n = \frac{x^k}{1 - x} \qquad |x| < 1 \tag{9.48}$$

we obtain Eq. (*9.20*); that is,

$$p_0 = \left[ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!} \sum_{n=s}^{\infty} \rho^n \right]^{-1} = \left[ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1 - \rho)} \right]^{-1}$$

provided $\rho = \lambda/(s\mu) < 1$.

**9.11.** Consider an M/M/*s* queueing system. Find the probability that an arriving customer is forced to join the queue.

An arriving customer is forced to join the queue when all servers are busy—that is, when the number of customers in the system is equal to or greater than *s*. Thus, using Eqs. (*9.20*) and (*9.21*), we get

$$P(\text{a customer is forced to join queue}) = \sum_{n=s}^{\infty} p_n = p_0 \frac{s^s}{s!} \sum_{n=s}^{\infty} \rho^n = p_0 \frac{(s\rho)^s}{s!(1 - \rho)}$$

$$= \frac{\dfrac{(s\rho)^s}{s!(1 - \rho)}}{\displaystyle\sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1 - \rho)}} \tag{9.49}$$

Equation (*9.49*) is sometimes referred to as *Erlang's delay* (or *C*) formula and denoted by $C(s, \lambda/\mu)$. Equation (*9.49*) is widely used in telephone systems and gives the probability that no trunk (server) is available for an incoming call (arriving customer) in a system of *s* trunks.

**9.12.** Derive Eqs. (*9.22*) and (*9.23*).

Equation (*9.21*) can be rewritten as

$$p_n = \begin{cases} \dfrac{(s\rho)^n}{n!}\, p_0 & n \le s \\ \dfrac{\rho^n s^s}{s!}\, p_0 & n > s \end{cases}$$

Then the average number of customers in the system is

$$\begin{aligned} L = E(N) &= \sum_{n=0}^{\infty} n p_n = p_0 \left[ \sum_{n=0}^{s} n \frac{(s\rho)^n}{n!} + \sum_{n=s+1}^{\infty} n \frac{\rho^n s^s}{s!} \right] \\ &= p_0 \left[ s\rho \sum_{n=1}^{s} \frac{(s\rho)^{n-1}}{(n-1)!} + \frac{s^s}{s!} \sum_{n=s+1}^{\infty} n\rho^n \right] \\ &= p_0 \left[ s\rho \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!} \left( \sum_{n=0}^{\infty} n\rho^n - \sum_{n=0}^{s} n\rho^n \right) \right] \end{aligned}$$

Using the summation formulas,

$$\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \qquad |x| < 1 \tag{9.50}$$

$$\sum_{n=0}^{k} n x^n = \frac{x[kx^{k+1} - (k+1)x^k + 1]}{(1-x)^2} \qquad |x| < 1 \tag{9.51}$$

and Eq. (*9.20*), we obtain

$$\begin{aligned} L &= p_0 \left( s\rho \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{s^s}{s!} \left\{ \frac{\rho}{(1-\rho)^2} - \frac{\rho[s\rho^{s+1} - (s+1)\rho^s + 1]}{(1-\rho)^2} \right\} \right) \\ &= p_0 \left\{ s\rho \left[ \sum_{n=0}^{s-1} \frac{(s\rho)^n}{n!} + \frac{(s\rho)^s}{s!(1-\rho)} \right] + \frac{\rho(\rho s)^s}{s!(1-\rho)^2} \right\} \\ &= p_0 \left[ s\rho \frac{1}{p_0} + \frac{\rho(s\rho)^s}{s!(1-\rho)^2} \right] \\ &= s\rho + \frac{\rho(s\rho)^s}{s!(1-\rho)^2}\, p_0 = \frac{\lambda}{\mu} + \frac{\rho(s\rho)^s}{s!(1-\rho)^2}\, p_0 \end{aligned}$$

Next, using Eqs. (*9.21*) and (*9.50*), the average number of customers in the queue is

$$\begin{aligned} L_q &= \sum_{n=s}^{\infty} (n-s) p_n = \sum_{n=s}^{\infty} (n-s) \frac{\rho^n s^s}{s!}\, p_0 \\ &= p_0 \frac{(s\rho)^s}{s!} \sum_{n=s}^{\infty} (n-s)\rho^{n-s} = p_0 \frac{(s\rho)^s}{s!} \sum_{k=0}^{\infty} k\rho^k \\ &= \frac{\rho(s\rho)^s}{s!(1-\rho)^2}\, p_0 = L - \frac{\lambda}{\mu} \end{aligned}$$

**9.13.** A corporate computing center has two computers of the same capacity. The jobs arriving at the center are of two types, internal jobs and external jobs. These jobs have Poisson arrival times with rates 18 and 15 per hour, respectively. The service time for a job is an exponential r.v. with mean 3 minutes.

(*a*) Find the average waiting time per job when one computer is used exclusively for internal jobs and the other for external jobs.

(*b*) Find the average waiting time per job when two computers handle both types of jobs.

(*a*) When the computers are used separately, we treat them as two M/M/1 queueing systems. Let $W_{q1}$ and $W_{q2}$ be the average waiting time per internal job and per external job, respectively. For internal jobs,

$\lambda_1 = \frac{18}{60} = \frac{3}{10}$ and $\mu_1 = \frac{1}{3}$. Then, from Eq. (9.16),

$$W_{q1} = \frac{\frac{3}{10}}{\frac{1}{3}(\frac{1}{3} - \frac{3}{10})} = 27 \text{ min}$$

For external jobs, $\lambda_2 = \frac{15}{60} = \frac{1}{4}$ and $\mu_2 = \frac{1}{3}$, and

$$W_{q2} = \frac{\frac{1}{4}}{\frac{1}{3}(\frac{1}{3} - \frac{1}{4})} = 9 \text{ min}$$

(*b*) When two computers handle both types of jobs, we model the computing service as an M/M/2 queueing system with

$$\lambda = \frac{18 + 15}{60} = \frac{11}{20}, \qquad \mu = \frac{1}{3} \qquad \rho = \frac{\lambda}{2\mu} = \frac{33}{40}$$

Now, substituting $s = 2$ in Eqs. (9.20), (9.22), (9.24), and (9.25), we get

$$p_0 = \left[1 + 2\rho + \frac{(2\rho)^2}{2(1-\rho)}\right]^{-1} = \frac{1-\rho}{1+\rho} \tag{9.52}$$

$$L = 2\rho + \frac{4\rho^3}{2(1-\rho)^2}\left(\frac{1-\rho}{1+\rho}\right) = \frac{2\rho}{1-\rho^2} \tag{9.53}$$

$$W_q = \frac{L}{\lambda} - \frac{1}{\mu} = \frac{1}{\lambda}\frac{2\rho}{1-\rho^2} - \frac{1}{\mu} \tag{9.54}$$

Thus, from Eq. (9.54), the average waiting time per job when both computers handle both types of jobs is given by

$$W_q = \frac{2(\frac{33}{40})}{\frac{11}{20}[1 - (\frac{33}{40})^2]} = 6.39 \text{ min}$$

From these results, we see that it is more efficient for both computers to handle both types of jobs.

**9.14.** Derive Eqs. (9.27) and (9.28).

From Eqs. (9.26) and (9.10), we have

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0 \qquad 0 \le n \le K \tag{9.55}$$

From Eq. (9.11), $p_0$ is obtained by equating

$$\sum_{n=0}^{K} p_n = p_0 \sum_{n=0}^{K} \rho^n = 1$$

Using the summation formula

$$\sum_{n=0}^{K} x^n = \frac{1 - x^{K+1}}{1 - x} \tag{9.56}$$

we obtain

$$p_0 = \frac{1-\rho}{1-\rho^{K+1}} \qquad \text{and} \qquad p_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}}$$

Note that in this case, there is no need to impose the condition that $\rho = \lambda/\mu < 1$.

**9.15.** Derive Eq. (9.30).

Using Eqs. (9.28) and (9.51), the average number of customers in the system is given by

$$L = E(N) = \sum_{n=0}^{K} np_n = \frac{1-\rho}{1-\rho^{K+1}} \sum_{n=0}^{K} n\rho^n$$

$$= \frac{1-\rho}{1-\rho^{K+1}} \left\{ \frac{\rho[K\rho^{K+1} - (K+1)\rho^K + 1]}{(1-\rho)^2} \right\}$$

$$= \rho \frac{1 - (K+1)\rho^K + K\rho^{K+1}}{(1-\rho)(1-\rho^{K+1})}$$

**9.16.** Consider the M/M/1/$K$ queueing system. Show that

$$L_q = L - (1 - p_0) \tag{9.57}$$

$$W_q = \frac{1}{\mu} L \tag{9.58}$$

$$W = \frac{1}{\mu}(L + 1) \tag{9.59}$$

In the M/M/1/$K$ queueing system, the average number of customers in the system is

$$L = E(N) = \sum_{n=0}^{K} np_n \qquad \text{and} \qquad \sum_{n=0}^{K} p_n = 1$$

The average number of customers in the queue is

$$L_q = E(N_q) = \sum_{n=1}^{K} (n-1)p_n = \sum_{n=0}^{K} np_n - \sum_{n=1}^{K} p_n = L - (1 - p_0)$$

A customer arriving with the queue in state $S_n$ has a wait time $T_q$ that is the sum of $n$ independent exponential r.v.'s, each with parameter $\mu$. The expected value of this sum is $n/\mu$ [Eq. (4.108)]. Thus, the average amount of time that a customer spends waiting in the queue is

$$W_q = E(T_q) = \sum_{n=0}^{K} \frac{n}{\mu} p_n = \frac{1}{\mu} \sum_{n=0}^{K} np_n = \frac{1}{\mu} L$$

Simililarly, the amount of time that a customer spends in the system is

$$W = E(T) = \sum_{n=1}^{K} \frac{(n+1)}{\mu} p_n = \frac{1}{\mu} \left( \sum_{n=0}^{K} np_n + \sum_{n=0}^{K} p_n \right) = \frac{1}{\mu}(L + 1)$$

Note that Eqs. (9.57) to (9.59) are equivalent to Eqs. (9.31) to (9.33) (Prob. 9.27).

**9.17.** Derive Eqs. (9.35) and (9.36).

As in Prob. 9.10, from Eqs. (9.34) and (9.10), we have

$$p_n = p_0 \prod_{k=0}^{n-1} \frac{\lambda}{(k+1)\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!} \qquad n < s \tag{9.60}$$

$$p_n = p_0 \prod_{k=0}^{s-1} \frac{\lambda}{(k+1)\mu} \sum_{k=s}^{n-1} \frac{\lambda}{s\mu} = p_0 \left(\frac{\lambda}{\mu}\right)^n \frac{1}{s!\, s^{n-s}} \qquad s \le n \le K \tag{9.61}$$

Let $\rho = \lambda/(s\mu)$. Then Eqs. (9.60) and (9.61) can be rewritten as

$$p_n = \begin{cases} \dfrac{(s\rho)^n}{n!} p_0 & n < s \\ \dfrac{\rho^n s^s}{s!} p_0 & s \le n \le K \end{cases}$$

which is Eq. (9.36). From Eq. (9.11), $p_0$ is obtained by equating

$$\sum_{n=0}^{K} p_n = p_0\left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \sum_{n=s}^{K}\frac{\rho^n s^s}{s!}\right] = 1$$

Using the summation formula (9.56), we obtain

$$\begin{aligned} p_0 &= \left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\sum_{n=s}^{K}\rho^n\right]^{-1} \\ &= \left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \frac{s^s}{s!}\left(\sum_{n=0}^{K}\rho^n - \sum_{n=0}^{s-1}\rho^n\right)\right]^{-1} \\ &= \left[\sum_{n=0}^{s-1}\frac{(s\rho)^n}{n!} + \frac{(s\rho)^s(1-\rho^{K-s+1})}{s!(1-\rho)}\right]^{-1} \end{aligned}$$

which is Eq. (9.35).

**9.18.** Derive Eq. (9.37).

Using Eq. (9.36) and (9.51), the average number of customers in the queue is given by

$$\begin{aligned} L_q &= \sum_{n=s}^{K}(n-s)p_n = p_0\frac{s^s}{s!}\sum_{n=s}^{K}(n-s)\rho^n \\ &= p_0\frac{(s\rho)^s}{s!}\sum_{n=s}^{K}(n-s)\rho^{n-s} = p_0\frac{(s\rho)^s}{s!}\sum_{m=0}^{K-s}m\rho^m \\ &= p_0\frac{(s\rho)^s}{s!}\frac{\rho[(K-s)\rho^{K-s+1} - (K-s+1)\rho^{K-s} + 1]}{(1-\rho)^2} \\ &= p_0\frac{\rho(s\rho)^s}{s!(1-\rho)^2}\{1 - [1 + (1-\rho)(K-s)]\rho^{K-s}\} \end{aligned}$$

**9.19.** Consider an M/M/$s$/$s$ queueing system. Find the probability that all servers are busy.

Setting $K = s$ in Eqs. (9.60) and (9.61), we get

$$p_n = p_0\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!} \qquad 0 \le n \le s \tag{9.62}$$

and $p_0$ is obtained by equating

$$\sum_{n=0}^{s} p_n = p_0\left[\sum_{n=0}^{s}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}\right] = 1$$

Thus

$$p_0 = \left[\sum_{n=0}^{s}\left(\frac{\lambda}{\mu}\right)^n\frac{1}{n!}\right]^{-1} \tag{9.63}$$

The probability that all servers are busy is given by

$$p_s = p_0\left(\frac{\lambda}{\mu}\right)^s\frac{1}{s!} = \frac{(\lambda/\mu)^s/s!}{\sum_{n=0}^{s}(\lambda/\mu)^n/n!} \tag{9.64}$$

Note that in an M/M/$s$/$s$ queueing system, if an arriving customer finds that all servers are busy, the customer will turn away and is lost. In a telephone system with $s$ trunks, $p_s$ is the portion of incoming calls which will receive a busy signal. Equation (9.64) is often referred to as *Erlang's loss* (or *B*) formula and is commonly denoted as $B(s, \lambda/\mu)$.

**9.20.** An air freight terminal has four loading docks on the main concourse. Any aircraft which arrive when all docks are full are diverted to docks on the back concourse. The average aircraft arrival

rate is 3 aircraft per hour. The average service time per aircraft is 2 hours on the main concourse and 3 hours on the back concourse.

(*a*) Find the percentage of the arriving aircraft that are diverted to the back concourse.

(*b*) If a holding area which can accommodate up to 8 aircraft is added to the main concourse, find the percentage of the arriving aircraft that are diverted to the back concourse and the expected delay time awaiting service.

(*a*) The service system at the main concourse can be modeled as an M/M/*s*/*s* queueing system with $s = 4$, $\lambda = 3$, $\mu = \frac{1}{2}$, and $\lambda/\mu = 6$. The percentage of the arriving aircraft that are diverted to the back concourse is

$$100 \times P(\text{all docks on the main concourse are full})$$

From Eq. (*9.64*),

$$P(\text{all docks on the main concourse are full}) = p_4 = \frac{6^4/4!}{\sum_{n=0}^{4} (6^n/n!)} = \frac{54}{115} \approx 0.47$$

Thus, the percentage of the arriving aircraft that are diverted to the back concourse is about 47 percent.

(*b*) With the addition of a holding area for 8 aircraft, the service system at the main concourse can now be modeled as an M/M/*s*/*K* queueing system with $s = 4$, $K = 12$, and $\rho = \lambda/(s\mu) = 1.5$. Now, from Eqs. (*9.35*) and (*9.36*),

$$p_0 = \left[ \sum_{n=0}^{3} \frac{6^n}{n!} + \frac{6^4}{4!} \left( \frac{1 - 1.5^9}{1 - 1.5} \right) \right]^{-1} \approx 0.00024$$

$$p_{12} = \frac{1.5^{12} 4^4}{4!} p_0 \approx 0.332$$

Thus, about 33.2 percent of the arriving aircraft will still be diverted to the back concourse.

Next, from Eq. (*9.37*), the average number of aircraft in the queue is

$$L_q = 0.00024 \frac{1.5(6^4)}{4!(1 - 1.5)^2} \{1 - [1 + (1 - 1.5)8](1.5)^8\} \approx 6.0565$$

Then, from Eq. (*9.40*), the expected delay time waiting for service is

$$W_q = \frac{L_q}{\lambda(1 - p_{12})} = \frac{6.0565}{3(1 - 0.332)} \approx 3.022 \text{ hours}$$

Note that when the 2-hour service time is added, the total expected processing time at the main concourse will be 5.022 hours compared to the 3-hour service time at the back concourse.

## Supplementary Problems

**9.21.** Customers arrive at the express checkout lane in a supermarket in a Poisson process with a rate of 15 per hour. The time to check out a customer is an exponential r.v. with mean of 2 minutes.

(*a*) Find the average number of customers present.

(*b*) What is the expected idle delay time experienced by a customer?

(*c*) What is the expected time for a customer to clear a system?

*Ans.* (*a*) 1; (*b*) 2 min; (*c*) 4 min

**9.22.** Consider an M/M/1 queueing system. Find the probability of finding at least $k$ customers in the system.

*Ans.* $\rho^k = (\lambda/\mu)^k$.

**9.23.** In a university computer center, 80 jobs an hour are submitted on the average. Assuming that the computer service is modeled as an M/M/1 queueing system, what should the service rate be if the average turnaround time (time at submission to time of getting job back) is to be less than 10 minutes?

*Ans.* 1.43 jobs per minute

**9.24.** The capacity of a communication line is 2000 bits per second. The line is used to transmit 8-bit characters, and the total volume of expected calls for transmission from many devices to be sent on the line is 12,000 characters per minute. Find (*a*) the traffic intensity, (*b*) the average number of characters waiting to be transmitted, and (*c*) the average transmission (including queueing delay) time per character.

*Ans.* (*a*) 0.8; (*b*) 3.2; (*c*) 20 ms

**9.25.** A bank counter is currently served by two tellers. Customers entering the bank join a single queue and go to the next available teller when they reach the head of the line. On the average, the service time for a customer is 3 minutes, and 15 customers enter the bank per hour. Assuming that the arrivals process is Poisson and the service time is an exponential r.v., find the probability that a customer entering the bank will have to wait for service.

*Ans.* 0.205

**9.26.** A post office has three clerks serving at the counter. Customers arrive on the average at the rate of 30 per hour, and arriving customers are asked to form a single queue. The average service time for each customer is 3 minutes. Assuming that the arrivals process is Poisson and the service time is an exponential r.v., find (*a*) the probability that all the clerks will be busy, (*b*) the average number of customers in the queue, and (*c*) the average length of time customers have to spend in the post office.

*Ans.* (*a*) 0.237; (*b*) 0.237; (*c*) 3.947 min

**9.27.** Show that Eqs. (*9.57*) to (*9.59*) and Eqs. (*9.31*) to (*9.33*) are equivalent.

*Hint:* Use Eq. (*9.29*).

**9.28.** Find the average number of customers $L$ in the M/M/1/$K$ queueing system when $\lambda = \mu$.

*Ans.* $K/2$

**9.29.** A gas station has one diesel fuel pump for trucks only and has room for three trucks (including one at the pump). On the average trucks arrive at the rate of 4 per hour, and each truck takes 10 minutes to service. Assume that the arrivals process is Poisson and the service time is an exponential r.v.

(*a*) What is the average time for a truck from entering to leaving the station?

(*b*) What is the average time for a truck to wait for service?

(*c*) What percentage of the truck traffic is being turned away?

*Ans.* (*a*) 20.15 min; (*b*) 10.14 min; (*c*) 12.3 percent

**9.30.** Consider the air freight terminal service of Prob. 9.20. How many additional docks are needed so that at least 80 percent of the arriving aircraft can be served in the main concourse with the addition of holding area?

*Ans.* 4

# Appendix A

## Normal Distribution

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\xi^2/2}\, d\xi$$

$$\Phi(-z) = 1 - \Phi(z)$$

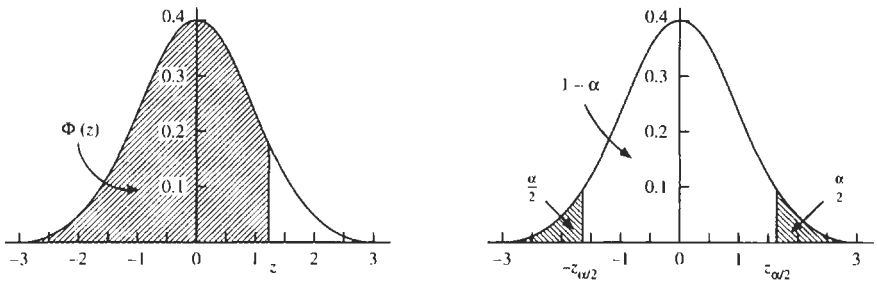



**Fig. A**

**Table A Normal Distribution $\Phi(z)$**

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5399 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7703 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8364 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |

**Table A—Continued**

| $z$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9078 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9085 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |
| 3.1 | 0.9990 | 0.9991 | 0.9991 | 0.9991 | 0.9992 | 0.9992 | 0.9992 | 0.9992 | 0.9993 | 0.9993 |
| 3.2 | 0.9993 | 0.9993 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9994 | 0.9995 | 0.9995 | 0.9995 |
| 3.3 | 0.9995 | 0.9995 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9996 | 0.9997 |
| 3.4 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9997 | 0.9998 | 0.9998 |
| 3.5 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 | 0.9998 |
| 3.6 | 0.9998 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 | 0.9999 |

The material below refers to Fig. A.

| $\alpha$ | 0.2 | 0.1 | 0.05 | 0.025 | 0.01 | 0.005 |
|---|---|---|---|---|---|---|
| $z_{\alpha/2}$ | 1.282 | 1.645 | 1.960 | 2.240 | 2.576 | 2.807 |

# Appendix B

# Fourier Transform

## B.1 CONTINUOUS-TIME FOURIER TRANSFORM

**Definition:**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

**Table B-1 Properties of the Continuous-Time Fourier Transform**

| Property | Signal | Fourier Transform |
|---|---|---|
| | $x(t)$ | $X(\omega)$ |
| | $x_1(t)$ | $X_1(\omega)$ |
| | $x_2(t)$ | $X_2(\omega)$ |
| Linearity | $a_1x_1(t) + a_2x_2(t)$ | $a_1X_1(\omega) + a_2X_2(\omega)$ |
| Time shifting | $x(t - t_0)$ | $e^{-j\omega t_0}X(\omega)$ |
| Frequency shifting | $e^{j\omega_0 t}x(t)$ | $X(\omega - \omega_0)$ |
| Time scaling | $x(at)$ | $\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$ |
| Time reversal | $x(-t)$ | $X(-\omega)$ |
| Duality | $X(t)$ | $2\pi x(-\omega)$ |
| Time differentiation | $\frac{dx(t)}{dt}$ | $j\omega X(\omega)$ |
| Frequency differentiation | $(-jt)x(t)$ | $\frac{dX(\omega)}{d\omega}$ |
| Integration | $\int_{-\infty}^{t} x(\tau)\,d\tau$ | $\pi X(0)\delta(\omega) + \frac{1}{j\omega}X(\omega)$ |
| Convolution | $x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau$ | $X_1(\omega)X_2(\omega)$ |
| Multiplication | $x_1(t)x_2(t)$ | $\frac{1}{2\pi}X_1(\omega) * X_2(\omega)$ $= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\lambda)X_2(\omega-\lambda)\,d\lambda$ |
| Real signal | $x(t) = x_e(t) + x_o(t)$ | $X(\omega) = A(\omega) + jB(\omega)$ $X(-\omega) = X^*(\omega)$ |
| Even component | $x_e(t)$ | $\mathrm{Re}\{X(\omega)\} = A(\omega)$ |
| Odd component | $x_o(t)$ | $j\,\mathrm{Im}\{X(\omega)\} = jB(\omega)$ |
| Parseval's theorem | $\int_{-\infty}^{\infty} \lvert x(t)\rvert^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2\,d\omega$ | |

**Table B-2 Common Continuous-Time Fourier Transform Pairs**

| | $x(t)$ | $X(\omega)$ |
|---|---|---|
| 1. | $\delta(t)$ | $1$ |
| 2. | $\delta(t - t_0)$ | $e^{-j\omega t_0}$ |
| 3. | $1$ | $2\pi\delta(\omega)$ |
| 4. | $e^{j\omega_0 t}$ | $2\pi\delta(\omega - \omega_0)$ |
| 5. | $\cos \omega_0 t$ | $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ |
| 6. | $\sin \omega_0 t$ | $-j\pi[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$ |
| 7. | $u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$ | $\pi\delta(\omega) + \dfrac{1}{j\omega}$ |
| 8. | $e^{-at}u(t) \quad a > 0$ | $\dfrac{1}{j\omega + a}$ |
| 9. | $te^{-at}u(t) \quad a > 0$ | $\dfrac{1}{(j\omega + a)^2}$ |
| 10. | $e^{-a\lvert t\rvert} \quad a > 0$ | $\dfrac{2a}{a^2 + \omega^2}$ |
| 11. | $\dfrac{1}{a^2 + t^2}$ | $e^{-a\lvert\omega\rvert}$ |
| 12. | $e^{-at^2} \quad a > 0$ | $\sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/4a}$ |
| 13. | $p_a(t) = \begin{cases} 1 & \lvert t\rvert < a \\ 0 & \lvert t\rvert > a \end{cases}$ | $2a\dfrac{\sin \omega a}{\omega a}$ |
| 14. | $\dfrac{\sin at}{\pi t}$ | $p_a(\omega) = \begin{cases} 1 & \lvert\omega\rvert < a \\ 0 & \lvert\omega\rvert > a \end{cases}$ |
| 15. | $\operatorname{sgn} t = \begin{cases} 1 & t > 0 \\ -1 & t < 0 \end{cases}$ | $\dfrac{2}{j\omega}$ |
| 16. | $\sum_{k=-\infty}^{\infty} \delta(t - kT)$ | $\omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0),\ \omega_0 = \dfrac{2\pi}{T}$ |

## B.2 DISCRETE-TIME FOURIER TRANSFORM

**Definition:**

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} \qquad x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)e^{j\Omega n}\, d\Omega$$

**Table B-3 Properties of the Discrete-Time Fourier Transform**

| Property | Sequence | Fourier Transform |
|---|---|---|
| | $x(n)$ | $X(\Omega)$ |
| | $x_1(n)$ | $X_1(\Omega)$ |
| | $x_2(n)$ | $X_2(\Omega)$ |
| Periodicity | $x(n)$ | $X(\Omega + 2\pi) = X(\Omega)$ |
| Linearity | $a_1x_1(n) + a_2x_2(n)$ | $a_1X_1(\Omega) + a_2X_2(\Omega)$ |
| Time shifting | $x(n - n_0)$ | $e^{-j\Omega n_0}X(\Omega)$ |

**Table B-3—Continued**

| Property | Sequence | Fourier Transform |
|---|---|---|
| Frequency shifting | $e^{j\Omega_0 n}x(n)$ | $X(\Omega - \Omega_0)$ |
| Time reversal | $x(-n)$ | $X(-\Omega)$ |
| Frequency differentiation | $nx(n)$ | $j\dfrac{dX(\Omega)}{d\Omega}$ |
| Accumulation | $\displaystyle\sum_{k=-\infty}^{n} x(k)$ | $\pi X(0)\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}} X(\Omega)$ |
| Convolution | $x_1(n) * x_2(n) = \displaystyle\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$ | $X_1(\Omega)X_2(\Omega)$ |
| Multiplication | $x_1(n)x_2(n)$ | $\dfrac{1}{2\pi} X_1(\Omega) \otimes X_2(\Omega)$ $= \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} X_1(\lambda)X_2(\Omega - \lambda)\, d\lambda$ |
| Real sequence | $x(n) = x_e(n) + x_o(n)$ | $X(\Omega) = A(\Omega) + jB(\Omega)$ $X(-\Omega) = X^*(\Omega)$ |
| Even component | $x_e(n)$ | $\mathrm{Re}\{X(\Omega)\} = A(\Omega)$ |
| Odd component | $x_o(n)$ | $j\,\mathrm{Im}\{X(\Omega)\} = jB(\Omega)$ |
| Parseval's theorem | $\displaystyle\sum_{n=-\infty}^{\infty} \lvert x(n)\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(\Omega)\rvert^2\, d\Omega$ | |

**Table B-4 Common Discrete-Time Fourier Transform Pairs**

| | $x[n]$ | $X(\Omega)$ |
|---|---|---|
| 1. | $\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$ | $1$ |
| 2. | $\delta(n - n_0)$ | $e^{-j\Omega n_0}$ |
| 3. | $x(n) = 1$ | $2\pi\delta(\Omega)$ |
| 4. | $e^{j\Omega_0 n}$ | $2\pi\delta(\Omega - \Omega_0)$ |
| 5. | $\cos \Omega_0 n$ | $\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$ |
| 6. | $\sin \Omega_0 n$ | $-j\pi[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$ |
| 7. | $u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$ | $\pi\delta(\Omega) + \dfrac{1}{1 - e^{-j\Omega}}$ |
| 8. | $a^n u(n) \quad \lvert a\rvert < 1$ | $\dfrac{1}{1 - ae^{-j\Omega}}$ |
| 9. | $(n+1)a^n u(n) \quad \lvert a\rvert < 1$ | $\dfrac{1}{(1 - ae^{-j\Omega})^2}$ |
| 10. | $a^{\lvert n\rvert} \quad \lvert a\rvert < 1$ | $\dfrac{1 - a^2}{1 - 2a\cos\Omega + a^2}$ |
| 11. | $x(n) = \begin{cases} 1 & \lvert n\rvert \leq N_1 \\ 0 & \lvert n\rvert > N_1 \end{cases}$ | $\dfrac{\sin[\Omega(N_1 + \frac{1}{2})]}{\sin(\Omega/2)}$ |
| 12. | $\dfrac{\sin Wn}{\pi n} \quad 0 < W < \pi$ | $X(\Omega) = \begin{cases} 1 & 0 \leq \lvert\Omega\rvert \leq W \\ 0 & W < \lvert\Omega\rvert \leq \pi \end{cases}$ |
| 13. | $\displaystyle\sum_{k=-\infty}^{\infty} \delta(n - kN_0)$ | $\Omega_0 \displaystyle\sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0) \qquad \Omega_0 = \frac{2\pi}{N_0}$ |

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